Numerical Solution of Fully Fuzzy Linear Dynamical System

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Abstract: Many physical phenomenon or real life application in the area of Engineering, Biology, Economics, Physics and many other fields can be modeled in the form of dynamical system. In the simplest form it would be linear system, however for such a system while measuring the parameters, if there is an error of possible type then such can be more realistically modeled in fuzzy setup.

In this paper we give and prove the result for numerically solving the fully fuzzy linear system by Euler's method.

Keywords: Fully fuzzy linear dynamical system, Fuzzy number, Euler's scheme.

1. Introduction:

Dynamical system is very important tool to model real world phenomenon. Inaccuracy may occur modeling the dynamical system in form of collecting parameters, experimental parts or estimating initial values. To remove this type of possibilistic uncertainty, fuzzy setup comes in to scenario. The development of Fuzzy Differential equations has been growing very rapidly in recent years.

Fuzzy derivative was first introduced by Chang and Zadeh in 1972 as in [1], followed by Dubois and Prade in 1982 refer [2]. The term "fuzzy differential equation" was first coined by Byatt and Kandel. Fuzzy differential equations with initial value problem were first simultaneously studied by Kaleva and Seikkala in 1987 as in [3, 4]. Initially the solution of fuzzy differential equation given by Hukuhara derivative but it has disadvantage it is only applicable for increasing support. Behavior of fuzzy solution is different from crisp solution. So Seikkala introduced new derivative which is modified form of Hukuhara derivative. Hüllermeier refer [5] gave fuzzy differential equation as family of differential inclusions. Bede and Gal gave the concept of strongly generalized differentiability as in [6] and studied by I.J Rudas, A.L Benscsik as in [7] and Y. Chalco- Cano, H Romàn Flores refer [8]. Allahviranloo et. al. used differential transform method by using generalized H–differentbility as in [9].

S. Abbasbandy, T. Allaviranloo, O. Lopez-Pouso, J.J. Nieto as in [10] proposed paper on Numerical methods for fuzzy differential inclusions. Maryam Mosleh refer [11] proposed Numerical solution of fuzzy differential equations under generalized differentiability by fuzzy neural network. In this paper he utilized the generalized characterization theorem. M.Otadi, M. Mosleh as in [12] proposed Numerical solution of hybrid fuzzy differential equations by fuzzy neural network. Sankar Prasad Mondal and Tapan kumar Roy as in [13] proposed first order linear non homogeneous ODE in fuzzy environment based on Laplace transform. They conclude this process can be applied for any biological and economical mathematical model. K. Ivaz, A. Khastan and Juan J. Nieto as in [14] proposed numerical method for fuzzy differential equations and hybrid fuzzy differential equation. They have shown the global error in Trapezoidal rule is much less than midpoint rules. For future research, they will apply Trapezoidal rule to fuzzy differential equations and hybrid fuzzy differential equations under generalized Hukuhara differentiability.

In this paper, we propose numerical technique to solve fully fuzzy dynamical system. After briefly describing the system initially in fuzzy setup, we give the preliminaries, and then establish the convergence of the proposed Euler scheme and in last section numerical illustration is given along with a real life application.

1.1 Fuzzy set up:

Consider a system of ODE in E^n with fuzzy coefficient,

$$\dot{X} = \tilde{A}X \tag{1}$$

Where $\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{bmatrix}$ is evolution matrix with fuzzy coefficient a_{ji}, and $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

With initial condition $X(0) = \tilde{x}_i^0 \ \forall\ i = 1,2,3, \dots \dots \dots \dots n$

where $a_{ji} = \left(a_{l_{ij}}, a_{m_{ij}}. a_{u_{ij}}\right)$ is a fuzzy positive triangular number, $\forall\ i, j = 1,2,3, \dots \dots n$

$$\dot{x}_i(t) = \textstyle\sum_{j=1}^{n} \widetilde{a_{ij}}\, x_j \qquad \forall\ i = 1,2,3, \dots \dots \dots \dots n$$

Now here we propose first order difference me

thod for the system (1).

(x_ij^(n + 1))̃

$$\tilde{x}_i^{n+1} = \tilde{x}_i^{\,n} + h\sum_{j=1}^{n}\widetilde{a_{ij}}\tilde{x}_j \qquad \forall\, i = 1,2,3, \dots \dots \dots \dots n \quad (2)$$

Equation (2) is a particular case of more general system,

$$\tilde{x}_i^{n+1} = \tilde{x}_i^{\,n} + h\,\widetilde{f_i}\,(t, \tilde{x}) \qquad (3)$$

Taking $\alpha\ cut$ on both side,

$$\left[\underline{x_i}^{n+1}, \overline{x_i}^{n+1}\right] = \left[\underline{x_i}^{n}, \overline{x_i}^{n}\right] + h\ \sum_{j=1}^{n}[\underline{a}_{ij}\ \overline{a}_{ij}]\ \ \left[\underline{x}_j, \overline{x}_j\right] \qquad \forall\, i = 1,2,3, \dots \dots \dots \dots n$$

Let $G_i = min\,(\overline{a}_{ij}\ \overline{x}_j, \underline{a}_{ij}\ \underline{x}_j, \overline{a}_{ij}\underline{x}_j, \underline{a}_{ij}\overline{x}_j)$, $H_i = max\ (\ \overline{a}_{ij}\ \ \overline{x}_j, \underline{a}_{ij}\ \underline{x}_j, \overline{a}_{ij}\ \underline{x}_j\ , \underline{a}_{ij}\ \overline{x}_j)$

$\forall\, i = 1,2,3, \dots \dots \dots \dots n$

$$\left[\underline{x_i}^{n+1}, \overline{x_i}^{n+1}\right] = \left[\underline{x_i}^{n}, \overline{x_i}^{n}\right] + h\ \sum_{j=1}^{n}\ [G_i, H_i]$$

Equating the upper and lower αcut, for all i= 1,2,3, … … … … n

$$\underline{x_i}^{n+1} = \underline{x_i}^{n} +\ h\sum_{j=1}^{n}\ G_i$$

$$\overline{x_i}^{n+1} = \overline{x_i}^{n} +\ h\sum_{j=1}^{n} H_i$$

For sake of convenience,

$$\widetilde{f_i}\,(t, x) = \sum_{j=1}^{n}\widetilde{a_{ij}}\widetilde{x_j} = [\underline{f_i}\ \ \overline{f_i}]$$

$$\overline{f_i}(x, t) = \sum_{j=1}^{n}\ H_i$$

$$\underline{f_i}(x, t) = \sum_{j=1}^{n}\ G_i$$

2. Preliminaries:

2.1. Fuzzy Number:

- A is a convex fuzzy set, i.e. $A(r\alpha + (1-\alpha)s) \geq \min\,[A(r),\ A(s)],\ \alpha\ \in [0,1]$ and $r, s \in X$;
- A is normal, i.e. $\exists x_0 \in X$ with $A(x_0) = 1$;
- A is upper semi-continuous i.e. $A(x_0) \geq \lim_{x\to x_0^{\pm}} A(x)$;
- $[A]^0 = \overline{sup\, p(A)} = \overline{\{x \in R | A(x) \geq 0\}}$ is compact, where $\bar{A}$ denotes the closure of A.

2.2. α-cut:

An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ordinary set $A^\alpha \subseteq X$, such that:

$$A^\alpha = \{x / ?_A (X) \geq \alpha\}\ \forall\, x \in X$$

2.3. Fuzzy Operation:

For $u, v \in R_f$ and $\lambda \in R$ the sum $u + v$ and the product $\lambda . u$ is defined as

$[u+v]^\alpha = [u]^\alpha + [v]^\alpha = [\overline{u}, \underline{u}] + [\overline{v}, \underline{v}] = [\underline{u} + \underline{v}, \overline{u} + \overline{v}]and[\lambda.u]^\alpha = \lambda[\overline{u}, \underline{u}] = [\lambda\overline{u}, \lambda\underline{u}]$ for all $\alpha \in [0,1]$

2.4. Generalized Hukuhara Derivative:

Consider a fuzzy mapping $F: (a,b) \to \mathcal{F}$ and $t_0 \in (a\,, b). F$ is strongly differentiable, at $t_0 \in (a, b)$ if there exist an element $F'(t_0) \in \mathcal{F}\ \forall\, h > 0$ such that

$$\lim_{h\to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h\to 0} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0)$$

or

$$\lim_{h\to 0} \frac{F(t_0 - h) \ominus F(t_0)}{-h} = \lim_{h\to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = F'(t_0)$$

or

$$\lim_{h\to 0} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h\to 0} \frac{F(t_0 - h) \ominus F(t_0 + h)}{-h} = F'(t_0)$$

or

$$\lim_{h\to 0} \frac{F(t_0) \ominus F(t_0 + h)}{-h} = \lim_{h\to 0} \frac{F(t_0) \ominus F(t_0 - h)}{-h} = F'(t_0)$$

2.5. First Decomposition Theorem:

For every $A \in \mathcal{F}$,

$$A = \bigcup_{\alpha \in [0\ 1]} {}_{\alpha}A$$

where ${}_{\alpha}A(x) = \alpha\ {}^{\alpha}A(x)$ and $\cup$ stands for fuzzy union.

3. Convergence of System:

To apply Euler scheme for system,

$$\tilde{x}_i^{n+1} = \tilde{x}_i^{\,n} + h\tilde{f}_i\,(t, x)$$

With initial condition $X(0) = \tilde{x}_i^{\,0}\ \forall\ i = 1,2,3, \dots \dots \dots \dots n$

Now we check the convergence of equation (3)

$$\tilde{x}_i^{n+1} = \tilde{x}_i^{\,n} + h\tilde{f}_i\,(t, x)$$

With initial condition $X(0) = x_i^{\,0}, \forall\ i = 1,2,3, \dots \dots \dots \dots n$

We use Lipchitz continuity as given in [15],

$$d\left[\tilde{f}\,(t, x), \tilde{f}\,(t, Y_i)\right] \le L_i\ d(\tilde{x}_i, \tilde{Y}_i)$$

$$L_i = \max \frac{\partial \tilde{f}_i\,(t, \tilde{x}_i)}{\partial \tilde{x}_i}$$

Before proving the convergence of scheme we give lemma,

Lemma: (Refer [16]), Let the sequence of numbers $\{e_n\}_{n=0}^{N}$ satisfy

$$|e_{n+1}| \le A\,|e_n| + B, \qquad 0 \le n\ \le N - 1$$

For the given positive constants A and B. Then

$$|e_n| \le\ A^n |e_0| + B\,\frac{A^n - 1}{A - 1},$$

Theorem:

Let $F = \tilde{f}_i(x, t)$ belong to $E^n(R)$, partial derivative of F is bounded over (R) and Lipchitz. Let $\widetilde{Y_i^n}$ be the exact solution and $\tilde{x}_i^{\,n}$ be the numerical solution of the system (1) by using Euler scheme then, the numerical solution converges to the exact solution $\tilde{Y}_i^{\,n}$ uniformly and is stable.

Proof:

Let $\widetilde{x}_i^{\,n}$ be the numerical solution of (1) by using Euler scheme given by equation (3) on the grid points.

Finally, defined error in *n*th iteration of *i*th component,

$$e_i^n = \tilde{Y}_i^n - \widetilde{x}_i^{\,n}$$

Consider, $$e_i^{n+1} = \tilde{Y}_i^{n+1} - \widetilde{x}_i^{\,n+1}$$

$$e_i^{n+1} = \tilde{Y}_i(t_n + h) - [\widetilde{x}_i^{\,n} + h(\widetilde{f}_i\,(t,x)\,]$$

By using Taylor's expansion and neglecting higher terms,

$$\left|e_i^{n+1}\right| = \left|\tilde{Y}_i(t_n) + h\tilde{Y}_i^{'}(t_n) + \frac{h^2\tilde{Y}_i^{"}(t_n + \theta h)}{2} - [\widetilde{x}_i^{\,n} + h(\widetilde{f}_i\,(t,x)\,]\right|$$

$$= \left|\tilde{Y}_i(t_n) - \widetilde{x}_i^{\,n} + h(\tilde{Y}_i^{'}(t_n) - \widetilde{f}_i\,(t,x)) + \frac{h^2\tilde{Y}_i^{"}(t_n + \theta h)}{2}\right|$$

$$=\left|\tilde{Y}_i(t_n) - \widetilde{x}_i^{\,n} + h(\widetilde{f}_i\,(t,Y) - \widetilde{f}_i\,(t,x)) + \frac{h^2\tilde{Y}_i^{"}(t_n + \theta h)}{2}\right|$$

Now by Lipschitz continuity,

$$=\left|\tilde{Y}_i(t_n) - \widetilde{x}_i^{\,n} + h(\widetilde{f}_i\,(t,Y) - \widetilde{f}_i\,(t,x)) + \frac{h^2\tilde{Y}_i^{"}(t_n + \theta h)}{2}\right|$$

$$=\left|\tilde{Y}_i(t_n) - \widetilde{x}_i^{\,n} + h(\tilde{Y}_i(t_n) - \widetilde{x}_i^{\,n})\frac{\partial \widetilde{f}_i\,(t,c)}{\partial x^i} + \frac{h^2\tilde{Y}_i^{"}(t_n + \theta h)}{2}\right|$$

$$\leq \left|\tilde{Y}_i(t_n) - \widetilde{x}_i^{\,n}\right|\left(1 + h\left|\frac{\partial \widetilde{f}_i\,(t,c)}{\partial x^i}\right|\right) + \frac{h^2\left|\tilde{Y}_i^{"}(t_n + \theta h)\right|}{2}$$

The error at *nth* iteration is given by,

$$\left|e_i^{n+1}\right| \leq \left|e_i^n\,(1 + hL_i) + M_i\frac{h^2}{2}\right|$$

Where M_i is the upper bound of $\left|\tilde{Y}_i^{"}(t_n + \theta h)\right|$.

$$\left|e_i^{n+1}\right| \leq \left|e_i^n\right|(1 + hL_i) + M_i\frac{h^2}{2}$$

Computing in backward manner we get at k^{th} step, $k<n$

$$|e_i^n| \leq \left|e_i^{n-1}\right| + h\, L_i\left|e_i^{n-1}\right| + M_i \frac{h^2}{2}$$

$$\left|e_i^{k-1}\right| \leq \left[\left|e_i^{k-2}\right|(1 + h\, L_i) + M_i \frac{h^2}{2}\right](1 + hL_i) + M_i \frac{h^2}{2}$$

$$\left|e_i^1\right| \leq \left|e_i^0\right| + h\, L_i\left|e_i^0\right| + M_i \frac{h^2}{2}$$

Application of lemma (1),

$$|e_i^n| \leq \left|e_i^0\right|(1 + hL_i)^n + M_i \frac{h^2}{2} \frac{((1 + hL_i)^n - 1)}{hL_i}$$

Let $A = 1 + hL_i$ & $B = M_i \frac{h^2}{2}$

$$|e_i^n| \leq \left|e_i^0\right|(A)^n + B \frac{((A)^n - 1)}{A - 1}$$

$$|e_i^n| \leq \left|e_i^0\right|(1 + hL_i)^n + M_i \frac{h^2}{2} \frac{((1 + hL_i)^n - 1)}{hL_i}$$

Suppose $e_0 = 0$ and $(1 + hL_i) \leq e^{hL_i}$

$$|e_i^n| \leq M_i \frac{h^2}{2} \frac{(e^{nhL_i} - 1)}{hL_i}$$

Which is valid for $0 \leq t_n = nh \leq T$ → stability, finally,

$$\leq M_i \frac{h^2}{2} \frac{(e^{TL_i} - 1)}{hL_i}$$

As $h \to 0$,

$$|e_i^n| \to 0$$

Now for all i, by taking $M = \max M_i$ and $L = \max L_i$

$$E_i \leq M \frac{h^2}{2} \frac{(e^{TL} - 1)}{hL} = M \frac{h}{2} \frac{(e^{TL} - 1)}{L} = O(h)$$

This establishes the convergence of proposed method.

Thus the solution of system (1) can be computed iteratively by given iterative formula (3).

In the next section we give the illustrative example to demonstrate the applicability of our proposed method for obtaining the solution of fully fuzzy linear dynamical system.

4. Numerical Example:

Consider an example,

$$\frac{dx_1}{dt} = \tilde{a}x_1 + \tilde{b}x_2$$
$$\frac{dx_2}{dt} = \tilde{c}x_1 + \tilde{d}x_2$$

where, the coefficients are triangular fuzzy numbers given as $\tilde{a} = (1,2,3)$, $\tilde{b} = (2,3,4)$, $\tilde{c} = (3,4,5)$, $\tilde{d} = (4,5,6)$, with, $\widetilde{x_1}^0 = (0.5,1,1.5)$, $\widetilde{x_2}^0 = (0.5,1,1.5)$ and $x \in R^2$

Using the proposed Euler scheme for solving the example we get for n > 0

$$\widetilde{x_1}^{n+1} = \widetilde{x_1}^n + h(\tilde{f}(t,\tilde{x})$$
$$\widetilde{x_2}^{n+1} = \widetilde{x_2}^n + h(\tilde{f}(t,\tilde{x})$$

Putting the values we get ,

$$\widetilde{x_1}^{n+1} = \widetilde{x_1}^n + h[(1,2,3)\widetilde{x_1}^n + (2,3,4)\widetilde{x_2}^n]$$
$$\widetilde{x_2}^{n+1} = \widetilde{x_2}^n + h\left[(3,4,5)\widetilde{x_1}^n + (4,5,6)\widetilde{x_2}^n\right]$$

The application of α cut on both the sides gives the general iterative scheme,

$$\left[\underline{x_1^{n+1}}, \overline{x_1^{n+1}}\right] = [\underline{x}_1^n, \overline{x}_1^n] + h\{ [1+\alpha, 3-\alpha][\underline{x}_1^n, \overline{x}_1^n] + [2+\alpha, 4-\alpha]\, [\underline{x}_2^n, \overline{x}_2^n]\}$$

$$\left[\underline{x_2^{n+1}}, \overline{x_2^{n+1}}\right] = \left[\underline{x}_2^n, \overline{x}_2^n\right] + h\,\{[3+\alpha, 5-\alpha][\underline{x}_1^n, \overline{x}_1^n] + [4+\alpha, 6-\alpha][\underline{x}_2^n, \overline{x}_2^n]\,\}$$

With initial condition

$$\tilde{x}_1{}^0 = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha], \tilde{x}_2{}^0 = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$$

Solving for n = 1 and taking step size *h*=0.01 we get,

$$\left[\underline{x}_1^1, \overline{x_1^1}\right] = [\underline{x}_1^0, \overline{x}_1^0] + 0.01\{ [1+\alpha, 3-\alpha][\underline{x}_1^0, \overline{x}_1^0] + [2+\alpha, 4-\alpha]\, [\underline{x}_2^0, \overline{x}_2^0]\}$$

$$\left[\underline{x}_2^1, \overline{x_2^1}\right] = \left[\underline{x}_2^0, \overline{x}_2^0\right] + h\,\{[3+\alpha, 5-\alpha][\underline{x}_2^0, \overline{x}_2^0] + [4+\alpha, 6-\alpha][\underline{x}_2^0, \overline{x}_2^0]\,\}$$

Put all the values,

$$\left[\underline{x_1^1}, \overline{x_1^1}\right] = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$$
$$+ 0.01\{[1 + \alpha, 3 - \alpha][0.5 + 0.5\alpha, 1.5 - 0.5\alpha] + [\, 2 + \alpha, 4 - \alpha]\ \ [0.5$$
$$+ 0.5\alpha, 1.5 - 0.5\alpha]\}$$

$$\left[\underline{x_1^1}, \overline{x_1^1}\right] = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha] + 0.01\{[0.5\alpha^2 + 0.5 + \alpha, 0.5\alpha^2 + 4.5 - 3\alpha]$$
$$+ [0.5\alpha^2 + 1 + 1.5\alpha, 0.5\alpha^2 + 6 - 3.5\alpha]\}$$

$$\left[\underline{x_1^1}, \overline{x_1^1}\right] = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha] + 0.01[\alpha^2 + 1.5 + 2.5\alpha, \alpha^2 + 10.5 - 6.5\alpha]$$

$$^{\alpha}x_1^1 = \left[\underline{x_1^1}, \overline{x_1^1}\right] = [0.01\alpha^2 + 0.525\alpha + 0.515, 0.01\alpha^2 - 0.565\alpha + 1.605]$$

Using first decomposition theorem x_1^1 can be constructed as,

$$x_1^1 = \begin{cases} -26.25 + 50\sqrt{0{,}25505 + 0.04x} & 0.515 \leq x \leq 1.05 \\ 28.25 \pm 50\sqrt{0.255025} + 0.04x & 1.05 \leq x \leq 1.605 \\ 0 & otherwise \end{cases}$$

Similarly,

$$\left[\underline{x_2^1}, \overline{x_2^1}\right] = [0.5 + 0.5\alpha, 1.5 - 0.5\alpha]$$
$$+\ 0.01\{[3 + \alpha, 5 - \alpha][0.5 + 0.5\alpha, 1.5 - 0.5\alpha], +[4 + \alpha, 6$$
$$- \alpha][0.5 + 0.5\alpha, 1.5 - 0.5\alpha]\}$$

$$^{\alpha}x_2^1 = \left[\underline{x_2^1}, \overline{x_2^1}\right] = [0.01\alpha^2 + 0.545\alpha + 0.535, 0.01\alpha^2 - 0.585\alpha + 1.665]$$

$$x_2^1 = \begin{cases} -27.25 + 50\sqrt{0.275625} + 0.04x & 0.535 \leq x \leq 1.09 \\ 29.25 \pm 50\sqrt{0.5184} + 0.04x & 1.09 \leq x \leq 1.665 \\ 0 & otherwise \end{cases}$$

Second iteration,

$$^{\alpha}x_1^2 = \left[\underline{x_1^2}, \overline{x_1^2}\right]=[0.0002\,\alpha^3 + 0.021\alpha^2 + 0.54165\alpha + 0.53085, -0.0002\,\alpha^3 + 0.0222\alpha^2 - 0.638005\alpha + 1.669\,]$$

$$^{\alpha}x_2^2 = \left[\underline{x_2^2}, \overline{x_2^2}\right]=[0.0002\,\alpha^3 + 0.0214\alpha^2 + 0.58805\alpha + 0.57185, -0.0002\,\alpha^3 + 0.0226\alpha^2 - 0.68105\alpha + 1.84515\,]$$

4.1. Real life problem:

Contamination of lakes and rivers is common problem. We can model this problem as dynamical system, while modeling exact estimation of quantity of toxic in discharge and fluid flow is not possible so the model in fuzzy setup gives more realistic depiction.

Consider the problem of predicting the impact of a toxic waste discharge into lakes and rivers. First, we construct dynamical system under some assumptions.

The toxic material leaks into the river at constant rate for one day. We assume that the toxic material mixes quickly and thoroughly with lake waters as it enters each lake. The toxic discharge will also have an impact on the second lake. Let water of lake 1 barely trickles into lake 2, suggests contamination of lake 2 should be minimal.

Let v_j denote constant volume of toxic material in lake *j*. Let τ_j denote the volume of toxic material in lake *j*. Let o_j denote the constant volume of liquid flow from lake *j* to river. Let i_j denote the constant liquid flow in each lake.

Since, the volume v_j is consumed: $i_1 = i_2 + o_1$ & $i_2 = o_2$

The concentration $c_j = \frac{\tau_j}{v_j}$

The concentration c_0 of toxic material in the water flowing into lake 1: $r(t) = c_0 i_1$

Assuming that liquid enters or leaves the river and lake system only by flowing downstream the amount of toxin in lake 1 is

$$\tau_1(t) = \int_0^t r(s)ds - \int_0^t c_1 i_2 ds - \int_0^t c_1 o_1 ds$$

And the amount in lake 2 is

$$\tau_2(t) = \int_0^t c_1 i_2 ds - \int_0^t c_2 o_2 ds$$

Differentiation leads to the equations

$$\frac{d\tau_1}{dt} = r(t) - c_1 i_2 - c_1 o_1$$

$$\frac{d\tau_2}{dt} = c_1 i_2 - c_2 o_2$$

Using the conservation equation $i_1 = i_2 + o_1$ and the relationship $c_j = \frac{\tau_j}{v_j}$ for $j = 1, 2$, the equations become

$$\frac{d\tau_1}{dt} = r(t) - \frac{i_2}{v_1}\tau_1 - \frac{o_1}{v_1}\tau_1 = r(t) - \frac{i_1}{v_1}\tau_1$$

$$\frac{d\tau_2}{dt} = \frac{i_2}{v_1}\tau_1 - \frac{o_2}{v_2}\tau_2$$

Putting numerical value,

$$\frac{d\tau_1}{dt} = \widetilde{100} - \widetilde{0.5}\,\tau_1$$

$$\frac{d\tau_2}{dt} = \widetilde{0.25}\,\tau_1 - \tilde{1}\,\tau_2$$

With initial conditions: $\tau_1^0(t) = \widetilde{50}$ & $\tau_2^0(t) = \widetilde{10}$ and $h = 0.1$

Applying the Euler scheme and we get,

$$\tau_1^1(t) = [-0.3\,\alpha^2 + 13.1\,\alpha + 44.7, -0.3\,\alpha^2 - 11.9\,\alpha + 69.7]$$

$$\tau_2^1(t) = [-\,0.2\alpha^2 + 6.9\alpha + 3.57, -\,0.2\,\alpha^2 - 6.1\,\alpha + 16.55]$$

The other iteration values at different times for τ_1 and τ_2 are as shown in figure 1 and figure 2 respectively.

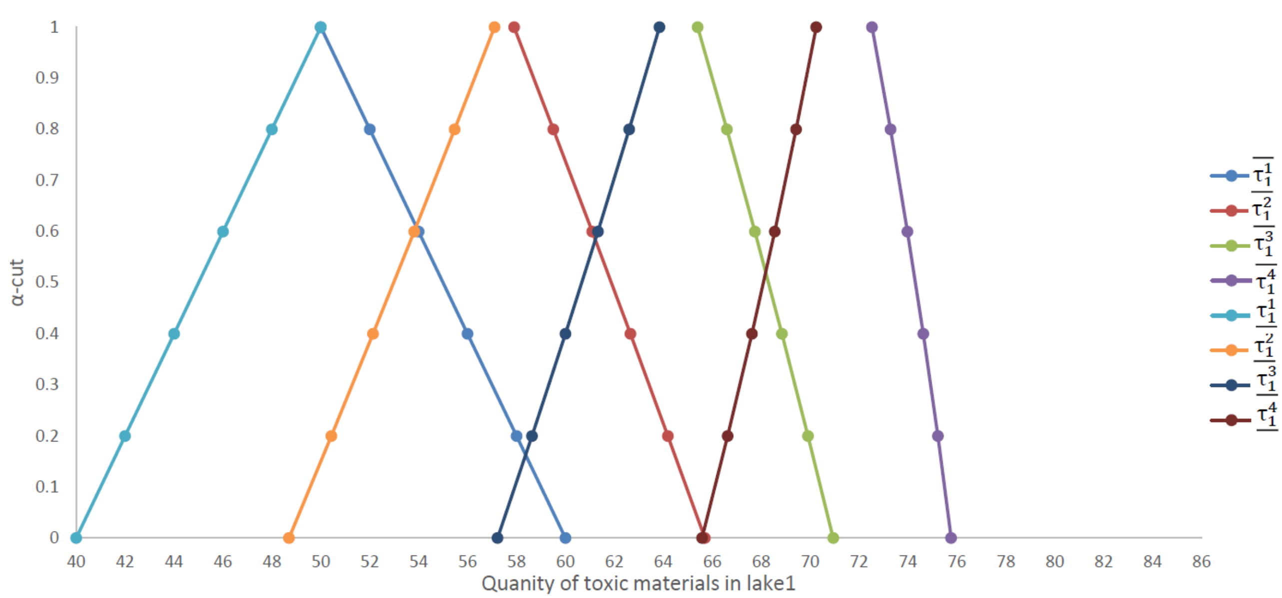

Fig. 1

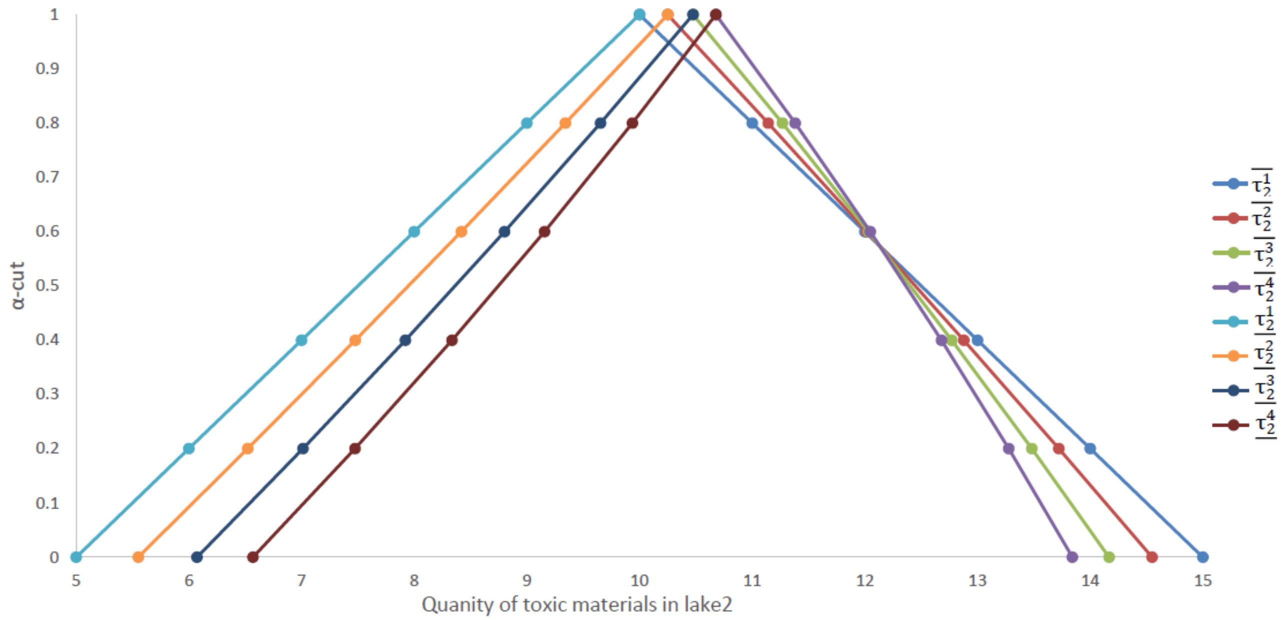

Fig. 2

We can see from the graphs the toxic materials discharged in both the lakes, shown as fuzzy numbers, as time increases.

5. Conclusion:

In our work, we propose Euler scheme for fully fuzzy linear dynamical system and establish its convergence. The illustrative example and the real life application of the proposed scheme is demonstrated. The solution of compartment system of river in which toxic material is discharged in some range is obtained using fuzzy numbers which is more realistic approach.

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