Chapter 6

Basis Criteria for Generalized Spline Modules on Some Isomorphic Graphs

6.1 Introduction

In this chapter, we have studied basis criteria for generalized splines on some isomorphic graphs over GCD domain.We observed that graphs which are isomorphic to each other have same or equivalent basis criteria since zero trails of these graphs are same and thus Q_G is also same for these graphs. We proved that basis criterion for generalized spline modules on each graph of an arbitrary set of isomorphic graphs is same over any GCD domain if flow-up basis exists for those graphs over GCD domain.

Thereafter, we focussed on the tree graphs as they are used abundantly in storing hierarchical data as non-linear data structures, organise data for quick search, insertion, deletion and network routing algorithms. Isomorphism in trees establish structural similarities between them, thereby finding variety of applications in diverse disciplines such as social sciences, science and technology, organisational structures etc. Subtree isomorphisms are used where pattern matching plays a significant role in searching the identical problems which can be compared. Defining algebraic structures such as rings and modules over trees provide deeper understanding of the underlying combinatorics thus opening newer areas of generating efficient algorithms for problem solving. We have seen that if two graphs are isomorphic then the zero trails of their vertices are identical.As a result we have concluded that isomorphic graphs have equivalent bases and basis criteria, whenever generalized modules on these graphs are free or generating sets exist.The result does not have strong implications for arbitrary graphs over GCD domains as no polynomial-time algorithm exists for checking the isomorphism between graphs in general.However,as AHU algorithm exists for tree graph isomorphisms, our results are generalized over ordered rooted tree graphs which form a very important class of graphs in Computer Network Theory.

Generalized spline modules on trees always have a flow-up basis even when R is not a PID[7].We have constructed flow-up basis for generalized spline modules on an arbitrary tree over any GCD domain. An algorithm is developed for indexing the vertices of an ordered rooted tree graph such that the above method can generate a flow-up basis for tree graph and its isomorphic graphs.

6.2 Preliminaries

Most of the results and discussions that will be needed to understand the results of this chapter are already discussed in the previous chapters.However,we mention some of them again in brief for the convenience of understanding the results of this chapter.

We first discuss the AHU algorithm as in [24], for the isomorphism in trees.

- **AHU Algorithm[24]**

  This Algorithm determines tree isomorphism in time $O(\mid V \mid)$ by associating a tuple with each vertex of a tree that describes the complete history of its descendants.

  The AHU [24] algorithm is a serialization technique for representing the vertices of a tree as unique string and is able to capture a complete history of a tree's degree spectrum and structure, ensuring a deterministic method of checking tree isomorphisms.In this algorithm, leaf nodes are assigned with (). Every time we move upwards, we combine, sort and wrap the parentheses.We can't process a node until we have processed all its children.For example, suppose we have a tree with a single parent and two leaf nodes. So we assign () to the leaves. When we move towards the parent node, we combine the parentheses of leaves like ()() and wrap it in another pair of parentheses like (()()) and assign it to the parent. This process continues iteratively until we reach the root node. [See Fig.6.1].

  We particularly consider the rooted trees because root of a tree represents the "start" of the data. The root of a tree can be any vertex, but if it has to be determined uniquely, the centre of the tree is the best choice. Another important information assigned to the rooted trees is the ordering of the children (from left to right). Such a tree is called an ordered rooted tree. The definition of isomorphism in ordered rooted trees is as follows:

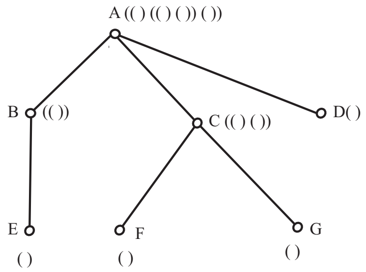

Fig. 6.1: Encoded ordered rooted tree using the AHU Algorithm

- **Isomorphism in ordered rooted trees[1]**

  Two ordered rooted trees are isomorphic if there exists an isomorphism of rooted trees, such that it preserves the order of children of every vertex. For example, the two trees in Fig.6.2 are isomorphic as trees but not as ordered rooted trees because the order of the children of the roots are different.

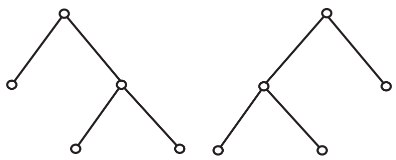

Fig. 6.2: Two isomorphic trees

We can encode any ordered rooted tree by assigning a string of 0's and 1's, which uniquely determine the tree by using the AHU algorithm. In AHU algorithm, parenthetical tuples are assigned to all tree vertices. However, these parenthetical tuples have no ordering. Replacing "(" with "1" and ")" with "0", the parenthetical names are converted into canonical names, which can be sorted lexicographically.

The following result from [24], can be used to check the isomorphism between two trees.

- **Theorem** [24]

  Two rooted trees are isomorphic if and only if they have the same canonical names assigned to the roots. An example of isomorphic trees with the same canonical names of the roots is shown in Fig.6.3.

The AHU -Tree isomorphism algorithm is $O(|V|^2)$, but can be improved to $O(|V|)$ by assigning and sorting canonical names by levels and checking by levels that canonical names agree.

As we know that Gilbert [55] has shown that the set of generalized splines $R_{(G,\alpha)}$ has a ring as well as a module structure over the base ring R. Also, definition 2.6 and proposition 2.7 [55] shows that isomorphism in graphs together with ring automorphism induces isomorphism between the ring of generalized splines.

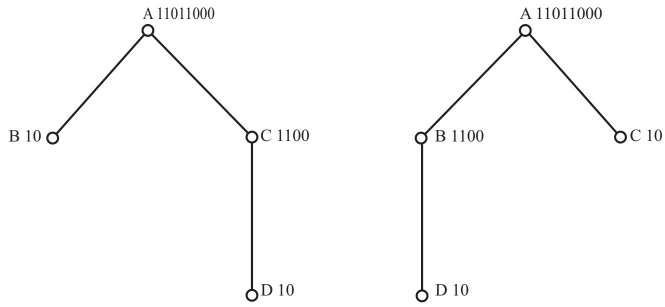

Fig. 6.3: Two isomorphic trees with same canonical names

We give the definition of isomorphism in edge labeled graphs which is a composition of ring automorphism together with graph isomorphisms as given in [55].It is as follows

- **Isomorphism in edge labeled graphs**[55]
  Let (G, α) and $(G^{'}, \alpha')$ be edge labelled graphs and R be a commutative ring. A homomorphism of edge labeled graphs $\phi : (G, \alpha) \longrightarrow (G^{'}, \alpha^{'})$ is a graph homomorphism $\phi_1 : G \longrightarrow G^{'}$ together with a ring automorphism $\phi_2 : R \longrightarrow R$, so that for each edge $e \in E_G$, we have $\phi_2(\alpha(e)) = \alpha^{'}(\phi_1(e))$.

$$\begin{tikzcd} E_G \arrow[r, "\Phi_1"] \arrow[d, "\alpha"'] & E_{G'} \arrow[d, "\alpha'"] \\ I \arrow[r, "\Phi_2"] & I \end{tikzcd}$$

An isomorphism of edge–labeled graphs is a homomorphism of edge–labeled graphs whose underlying graph homomorphism is an isomorphism.

- **Proposition**[55]**:**
  If $\phi : (G, \alpha) \longrightarrow (G^{'}, \alpha^{'})$ is an isomorphism of edge–labeled graphs then ϕ induces an isomorphism of the corresponding rings of generalized splines $\phi_* : R_G \longrightarrow R_{G'}$ defined as $\phi_*(p)_{\phi_1(u)} = \phi_2(p_u)$, for each u $\in V_G$. Using the above proposition, we have shown that isomorphic graphs together with ring automorphism also induce module isomorphism ϕ^* between $R_{(G,\alpha)}$ and $R_{(G',\alpha')}$.

The main result which was used by Selma Altinok and Samet Sarioglan in [8], for giving a necessary and sufficient condition for the existence of a basis for the module R_{C_n} is discussed below. First we discuss the formula for Q_G, for a graph G [defined in [8]], as follows:

Let (G,α) be an edge labeled graph with k vertices. Fix a vertex v_i on (G,α) with i $\geq$ 2. Label all vertices v_j with j $<$ i by zero. With notations in [8], Q_G is defined as

$Q_G = \prod_{i=2}^{k}$ [{($p_t^{(i,0)}$) } | t = 1,...,m_i] where m_i is the number of the zero trails of v_i. The above definition of Q_G gives a necessary and sufficient condition for the existence of basis for the generalized spline modules over the cycle graph C_n and tree graph as in [8].

Lemma 6.2.3 and 6.2.5 in [8] give formulae Q_G for cycle graph C_n and tree graph .Theorem 6.2.4 and 6.2.6 in [8] give the basis criteria for cycle graph and tree graph in terms of the determinant of flow-up classes $\{F_1, F_2, \ldots, F_n\}$ for these graphs.

- **Lemma**[8]

  Let (C_n, α) be an edge labeled n-cycle. Then $Q_{C_n} = \dfrac{l_1 l_2 \ldots l_n}{(l_1, l_2, \ldots, l_n)}$

- **Theorem** [8]

  Let (C_n,α) be an edge labeled n-cycle and let $\{F_1, \ldots, F_n\} \subset R_{(C_n,\alpha)}$. Then $\{F_1, \ldots, F_n\}$ forms a basis for $R_{(C_n,\alpha)}$ if and only if $| F_1 F_2 \ldots F_n |$ = r$\cdot Q_{C_n}$, where r $\in$ R is a unit.

- **Lemma** [8]

  Let G be a tree with n vertices and k edges. Then $Q_G = l_1 \ldots l_k$.

- **Theorem** [8]

  Let G be a tree with n vertices and k edges. Then $\{F_1, \ldots, F_n\} \subset R_{(G,\alpha)}$ forms a basis for $R_{(G,\alpha)}$ if and only if $| F_1 F_2 \ldots F_n |$=r$\cdot Q_G$, where $r \in R$ is a unit and R is a GCD domain.

6.3 Results and Discussions

In this section, first we extend the ring isomorphism $\phi_* : R_{(G,\alpha)} \longrightarrow R_{(G',\alpha')}$ to module isomorphism $\phi^* : R_{(G,\alpha)} \longrightarrow R_{(G',\alpha')}$ defined as follows:

$$\begin{aligned} \phi^*(rp)_{(\phi(u))} &= \phi_2(rp)_u \\ &= \phi_2(r(p)_u) \\ &= r(\phi_2(p_u)) \end{aligned}$$

for each $u \in V_G$

This is true for any base ring R over which the module $R_{(G,\alpha)}$ is defined. Thus, we see that if two graphs G and $G^{'}$ are isomorphic, then any ring automorphism on R induces a ring isomorphism $\phi_* : R_{(G,\alpha)} \longrightarrow R_{(G',\alpha')}$ and module isomorphism $\phi^* : R_{(G,\alpha)} \longrightarrow R_{(G',\alpha')}$.

It is proved in [55] by Gilbert, Polster and Tymoczko, that if R is a domain then the rank of the module $R_{(G,\alpha)}$ is equal to $\mid V \mid$. In this case the module $R_{(G',\alpha')}$ will also have the same rank $\mid V' \mid$, which is equal to $\mid V \mid$, and any basis B of $R_{(G,\alpha)}$ induces a basis B' of $R_{(G',\alpha')}$. Over a principal ideal domain R, the smallest flow - up classes exist over an arbitrary graph G [55], which forms a module basis for $R_{(G,\alpha)}$ and hence will induce module basis for $R_{(G',\alpha')}$, for an isomorphic graph G'.

We now consider the generalized spline modules $R_{(D_{3,3},\alpha)}$ and $R_{(D,\alpha')}$ for the diamond graph $D_{3,3}$ and its isomorphic graph D (Fig 6.4(a),6.4(b)), over a principal ideal domain R. If $F_1 = (1,1,1,1)$,$F_2 = (0, g_2, g_3, g_4)$,$F_3 = (0, 0, g_3, g_4)$,$F_4 = (0, 0, 0, g_4)$ forms a flow-up basis for $R_{(D_{3,3},\alpha)}$, the smallest leading entries g_2,g_3 and g_4 of F_2,F_3,and F_4 can be calculated by using the zero trails of the vertices v_2,v_3 and v_4 respectively ,as in [8].

Since the zero trails of the vertices are invariant under the graph isomorphism, the basis $\{F_1, F_2, F_3, F_4\}$ of $R_{(D_{3,3},\alpha)}$ will generate the basis $\{F'_1, F'_2, F'_3, F'_4\}$ of $R_{(D,\alpha')}$ as $Q_{D_{3,3}}$ is equal to Q_D by lemma 3.15[8].

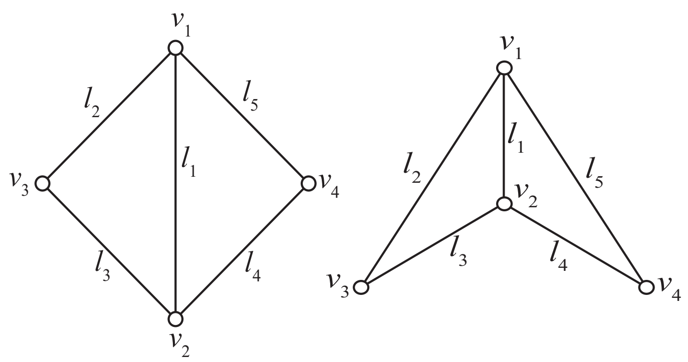

Fig. 6.4: a)Diamond graph $D_{3,3}$ and b)Isomorphic Graph D

Next, we give the basis criterion for generalized spline modules on a set of isomorphic graphs, over any GCD domain R.

- **Theorem**

  Let $\{G_1, G_2, \ldots, G_k\}$ be a set of isomorphic graphs. Then the basis criterion for generalized spline modules on each of these graphs, if exists, is same over any GCD domain.

  Proof From the definition of isomorphic graphs, each graph in the set $\{G_1, G_2, \ldots, G_k\}$ contains the same number of vertices connected in the same way. Therefore, from the definition of zero trail [Refer Chapter 2] of a particular vertex in G_i, for any i will be equal to the zero trail of the corresponding vertices in G_j, for all j. Hence, we have the set of smallest leading entries of all the flow-up splines in the graph G_i to be equal to the set of smallest leading entries of the flow up splines in the graphs G_j, for all j. From the definition of Q_G , if the set of smallest leading entries of the flow-up splines for $R_{(G_i,\alpha_i)}$ are equal to the set of smallest leading entries of the flow-up splines $R_{(G_j,\alpha_j)}$, for all j, Q_G will also be same for both the graphs. We conclude that the basis criterion for all the isomorphic graphs is same over any GCD domain.

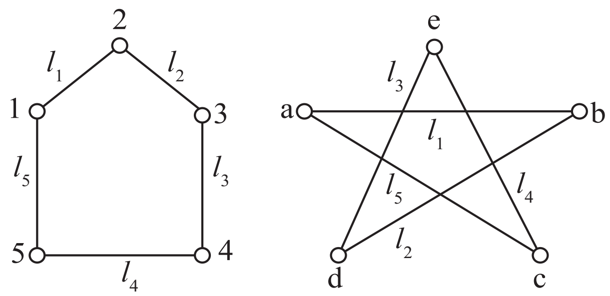

Fig. 6.5: (a)Cycle graph C_5 (b)Isomorphic graph C_5'

Considering Fig 6.5(a),6.5(b),we have two graphs, the cycle graph C_5 and it's isomorphic graph $C_5^{'}$.Since the zero trails of the vertices of both the graphs are same,they have same Q_G,defined as, $Q_G = \dfrac{(l_1 l_2 l_3 l_4 l_5)}{(l_1, l_2, l_3, l_4, l_5)}$, as calculated using the zero trails.

From the basis criteria,it follows that the set $\{F_1, F_2, F_3, F_4, F_5\} \in R_{(C_5,\alpha)}$ forms a basis if and only if $\mid F_1F_2F_3F_4F_5 \mid = rQ_{C_5}$, where $r \in R$ is a unit[8]. Since, $C_5^{'}$ is isomorphic to C_5, Q_{C_5} will be the same as $Q_{C_5'}$, and hence the images $\{F_1', F_2', F_3', F_4', F_5'\}$ will form a basis for $R_{(C_5',\alpha')}$ if and only if $\mid F_1'F_2'F_3'F_4'F_5' \mid = rQ_{C_5'}$,where $r \in R$.Thus the isomorphic graphs,C_5 and $C_5^{'}$,have same basis criteria over GCD domain R.

It was shown in [8], the generalized spline modules over trees are always free with the rank equal to the number of vertices, for any commutative ring R and a flow up basis exists for all such modules. Isomorphic trees are structurally identical and in turn induce isomorphism between the respective generalized spline modules. Thus, if two trees are isomorphic, then a correspondence is induced between the generating sets of their generalized spline modules and between the underlying database structures represented by them can be considered similar. We now use the zero-trail method to construct the flow-up basis for an ordered rooted tree. But before that, we first introduce level wise indexing of the nodes as follows: The root of the tree is indexed as "0". All the nodes in level one are the children of the root "0". Let there be n_0 vertices in level one, which are indexed as $01, 02, \ldots, 0n_0$ from left to right. Thus, the vertices in each level are ordered from left to right. Again, let the vertices $01, 02, \ldots, 0n_0$ have $n_{01}, n_{02}, \ldots, n_{0n_0}$ children respectively. We index the children of the vertex "01" as $011, 012, \ldots, 01n_{01}$ from left to right, the children of vertex "02" as $021, 022, \ldots, 02n_{02}$ from left to right and so on. Then the children of the node "$0n_0$" will be indexed as $0n_01, 0n_02, \ldots, 0n_0n_{0n_0}$ from left to right. Fig. 6.3 represents the level-wise ordered indexing of the nodes of an arbitrary tree.

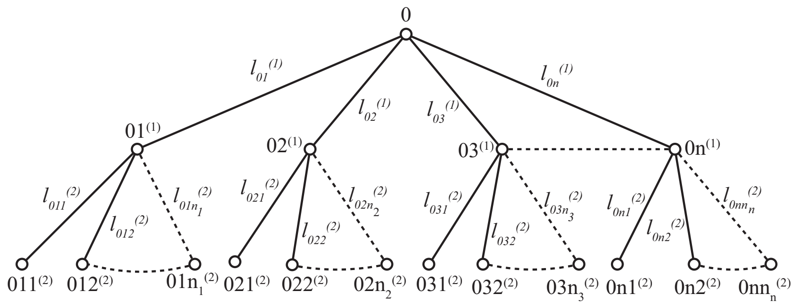

FIG. 6.6: Level-wise ordered indexing of the nodes of an arbitrary tree.

The edge label of the edge (u, v) can be represented as $l_v \in R$, where $l_v$is the generator of the ideal (l_v) associated with the edge (u, v) and v is a child of the vertex u. Thus, the edge ideal associated with the edge (0, 01) can be represented by the element l_{01} (Refer Fig.6.6) and the remaining edge labels can be represented in the same manner.

We see the example of star tree and ordered rooted tree with 7 vertices which are very important trees in database structures. Also, a similar algorithm can be generated for caterpillar, super caterpillar and lobster trees. The vertices of all these graphs are indexed as described before, starting from the root vertex indexed as "0".

First, we define the flow up classes of generalized splines for the star graph with six vertices(Fig. 6.7), using the zero trail method.

- **Star Tree with 6 vertices**

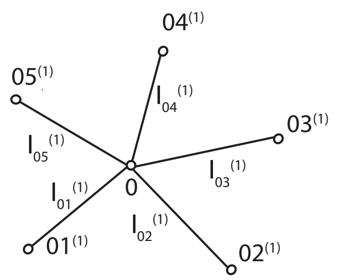

FIG. 6.7: Star Tree with 6 vertices

The indexing of the vertices is done level-wise as discussed before. The root vertex is indexed as 0, and all the five leaf vertices in level 1 are indexed as 01(1), 02(1), 03(1), 04(1) and 05(1) respectively. Thus, any generalized spline over this graph can be expressed as

$$P = \begin{bmatrix} p_{05}^{(1)} \\ p_{04}^{(1)} \\ p_{03}^{(1)} \\ p_{02}^{(1)} \\ p_{01}^{(1)} \\ p_0 \end{bmatrix}$$

Here $p_v \in R$ is the vertex label corresponding to the v^{th} vertex in the graph. The flow-up classes for this graph $\{F^0, F^{01(1)}, F^{02(1)}, F^{03(1)}, F^{04(1)}, F^{05(1)}\}$ are obtained as follows

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ l_{01}^{(1)} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ l_{02}^{(1)} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l_{03}^{(1)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ l_{04}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} l_{05}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Clearly, each class in the set $\{F^0, F^{01(1)}, F^{02(1)}, F^{03(1)}, F^{04(1)}, F^{05(1)}$ satisfies the GKM condition and hence is a generalized spline over the star graph with six vertices. Also, we can see that the determinant

$\mid F^0 \; F^{01(1)} \; F^{02(1)} \; F^{03(1)} \; F^{04(1)} \; F^{05(1)} \mid = l_{01}^{(1)} \; l_{02}^{(1)} \; l_{03}^{(1)} \; l_{04}^{(1)} \; l_{05}^{(1)} = Q_G$, where G is the star graph with six vertices in this case.

Hence, we conclude from theorem [2.14] in [8] that the set

$\{F^0 \; F^{01(1)} \; F^{02(1)} \; F^{03(1)} \; F^{04(1)} \; F^{05(1)} \}$ forms a basis for the generalized spline module $R_{(G,\alpha)}$ for this graph.

Next, we generalize the above method to arbitrary rooted tree graphs in which the vertices are ordered from left to right at all levels. Consider the ordered rooted tree with seven vertices as shown in Fig.6.8.

The root vertex is indexed as 0. There are two vertices in level 1, which are indexed as 01(1) and 02(1) , ordered from left to right. The four leaf vertices are indexed as 011(2), 012(2) (children of vertex 01(1)) and 021(2), 022(2)(children of vertex 02(1)),with the left to right ordering.

Any generalized spline over this graph can be expressed as

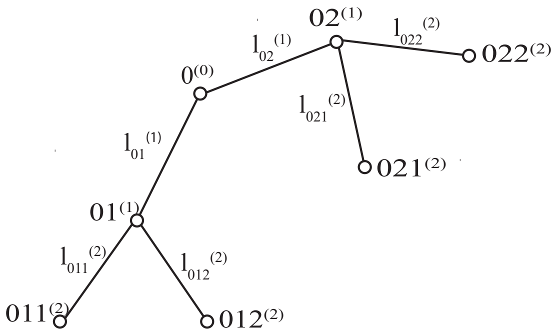

FIG. 6.8: Ordered rooted tree with 7 vertices

$$P = \begin{bmatrix} p^{(2)}_{022} \\ p^{(2)}_{021} \\ p^{(2)}_{012} \\ p^{(2)}_{011} \\ p^{(1)}_{02} \\ p^{(1)}_{01} \\ p_0 \end{bmatrix}$$

Using the zero trail method, we get the flow up classes of generalized splines for this graph as $\{F^0, F^{01(1)}, F^{02(1)}, F^{011(2)}, F^{012(2)}, F^{021(2)}, F^{022(2)}\}$ which is equal to

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l^{(1)}_{01} \\ l^{(1)}_{01} \\ 0 \\ l^{(1)}_{01} \\ 0 \end{pmatrix} \begin{pmatrix} l^{(1)}_{02} \\ l^{(1)}_{02} \\ 0 \\ 0 \\ l^{(1)}_{02} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ l^{(2)}_{011} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l^{(2)}_{012} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ l^{(2)}_{021} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} l^{(2)}_{022} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We see that the determinant of the matrix

$\mid F^0 F^{01(1)} F^{02(1)} F^{011(2)} F^{012(2)} F^{021(2)} F^{022(2)} \mid = l^{(1)}_{01} l^{(1)}_{02} l^{(2)}_{011} l^{(2)}_{012} l^{(2)}_{021} l^{(2)}_{022} = Q_G$, for the graph G.

Thus, the set of generalized splines $\{F^0 F^{01(1)} F^{02(1)} F^{011(2)} F^{012(2)} F^{021(2)} F^{022(2)}\}$ forms a basis for the module $R_{(G,\alpha)}$.

The algorithm for writing down the flow up basis for an arbitrary tree which is rooted and it's vertices at level are ordered from left to right is as follows

- All entries for the flow up basis element F^0 are one.
- Let there be n vertices in level 1, indexed as $01^{(1)}$, $02^{(1)}$,. . ., $0n^{(1)}$. The ordering of these vertices are taken from left to right. Then the corresponding elements

of flow up basis are $F^{01^{(1)}}$,$F^{02^{(1)}}$,..., $F^{0n^{(1)}}$, where $F^{0i^{(1)}}$ for $1 \leq i \leq n$ is constructed by taking $F_{0i}^{0i^{(1)}} = F_{0i1}^{0i^{(1)}} = F_{0i2}^{0i^{(1)}} = \ldots = F_{0in_i}^{0i^{(1)}} = l_{0i}^{(1)}$, and all other entries as zero. Here $F_v^{0i^{(1)}}$ denotes the vertex label of the vertex v in the generalized spline $F^{0i^{(1)}}$ and $0i1, 0i2, \ldots, 0in_i$, are the children of the vertex 0i. This construction ensures that the GKM conditions are satisfied by all the vertex labels of the spline $F^{0i^{(1)}}$.

– Similarly, the basis elements of the flow up basis corresponding to the children of the higher level vertices are constructed till we reach the leaf vertices. The leaf vertices will have only one non zero entry equal to the edge label of their parent vertices and zero otherwise.

It can be easily seen that the determinant of the matrix whose columns are the splines F^0,$F^{01^{(1)}}$,$F^{02^{(1)}}$,... is equal to the product of the edge labels and hence equal to Q_G, for the tree graph G. Thus the set of generalized splines $\{F^0, F^{01^{(1)}}, F^{02^{(1)}}, \ldots\}$ forms a flow up basis for G.

It can be seen that the level wise ordering of the vertices introduced in this paper, ensures that an algorithm can be developed for writing the flow up basis for any ordered rooted tree with a finite number of vertices.

6.4 Conclusions

We showed that generalized spline modules on isomorphic graphs over PID have same flow-up bases. We extended this result to generalized spline modules on isomorphic trees over any GCD domain and constructed Flow up basis for generalized spline modules on a star graph. An algorithm is developed for indexing the vertices of ordered rooted trees which helps us to generalize the method of constructing flow-up basis for generalized spline modules over any ordered rooted tree and hence over a family of isomorphic trees over a GCD domain. As rooted tree structures find vast applications in network analysis, manipulating hierarchical data, information searching, router algorithms, multi-stage decision-making and spread of infectious diseases, our study adds newer dimension to the existing knowledge and opens areas of further investigations.