Chapter 7

Research Summary and Conclusions

7.1 Research Statement

An edge labeled graph is a graph G whose edges are labeled with non-zero ideals of a commutative ring R. A Generalized Spline on an edge labeled graph G is a vertex labeling of G by elements of the ring R, such that the difference between any two adjacent vertex labels belongs to the ideal corresponding to the edge joining both the vertices. The set of generalized splines forms a sub ring of the product ring $R^{|V|}$, with respect to the operations of coordinate-wise addition and multiplication and also becomes a module over the ring R.This ring which is also a module is known as the generalized spline ring R_G, defined on the edge labeled graph G, for the commutative ring R. We have considered particular graphs such as complete graphs, complete bipartite graphs and hypercubes, labeling the edges with the non-zero ideals of an integral domain R and have identified the generalized spline ring R_G for these graphs. Also, general algorithms have been developed to find these splines for the above mentioned graphs, for any number of vertices and Python code has been written for finding these splines.We also determine conditions for a subset of $R_{(G,\alpha)}$ to form a basis for the spline module $R_{(G,\alpha)}$, for some classes of graphs such as Dutch Windmill graph and it's special cases such as friendship graph,butterfly graph over GCD domain.We find a generating set of flow-up classes for wheel graphs over the ring $\mathbb{Z}/p^k\mathbb{Z}$, where p is prime. Also we classify splines on cycles and wheel graphs over the ring $\mathbb{Z}/m\mathbb{Z}$ when m has few prime factors and find a generating set of flow-up classes on these graphs over $\mathbb{Z}/m\mathbb{Z}$. We also determine conditions for a subset of $R_{(G,\alpha)}$ to form a basis of $R_{(G,\alpha)}$ for some classes of graphs.We have studied basis criteria for generalized splines on some isomorphic graphs over GCD domain and constructed flow-up basis for generalized spline modules on an arbitrary tree.

7.2 Introduction

As Spline thoery started with Schoenberg's work in 1940's, he is considered as the pioneer of splines, which has now become a vast field in mathematics, finding extensive applications.To begin with, we give the classical definition of splines.

Let P be a d–dimensional polyhedral complex. A C^r–spline on P is a piecewise polynomial function (a polynomial is assigned to each d-dimensional cell or face σ of P), such that two polynomials supported on d-faces which share a common (d-1)-face τ, meet with order of smoothness r along the common face.The set of splines of degree at most k and are of smoothness of order r is denoted by $C^r_k(P)$, is a vector space[17].A C^r-spline is represented as a vector of polynomials $(f_1, f_2, \ldots, f_n)$, where each f_i is a polynomial of degree atmost k. Multiplying the vector by a fixed polynomial f gives $(f.f_1, f.f_2, \ldots, f.f_n)$, which is again a C^r-spline. This means that the set of splines is a module over the polynomial ring as shown in [17]. For two d-cells σ_1 and σ_2 sharing a common (d-1)–face τ , let l_τ be a nonzero linear form vanishing on τ. Billera and Rose [17] have shown that a pair of polynomials f_i supported on σ_i, $i = 1, 2$ meet with smoothness of order r along τ iff $l_\tau^{r+1} | f_1 - f_2$.

As an example, we see a 2-dimensional polyhedral complex which is a planar simplicial complex P and is the star of a single interior vertex v_0, the origin [Fig.7.1].The adjacent triangles or 2-faces meet over common lines, i.e, the 1- dimensional faces.

- **Example**[17]

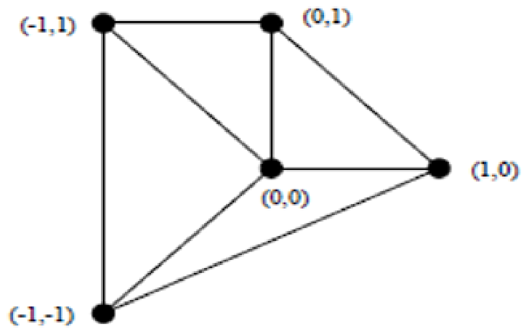


FIG. 7.1: Example of a C^r–spline

Begining with the simplex [Fig.7.1]in the first quadrant and moving clockwise, the piecewise polynomials defined on the triangles are f_1, f_2, f_3, f_4.To obtain a global C^r function, we require element (f_1, f_2, f_3, f_4) to satisfy the conditions, $a_1.y^{r+1} = f_1 - f_2; a_2.(x - y)^{r+1} = f_2 - f_3; a_3.(x + y)^{r+1} = f_3 - f_4; a_4.x^{r+1} = f_4 - f_1$. It can easily be verified that the continuity conditions are satisfied at every edge sharing the boundary of two simplexes[17].

Initially, piecewise polynomials were studied for curve-fitting and in fact, the word spline was used for particular C^2 interpolatory piecewise cubic polynomials. Later the definition was broadened to include any piecewise polynomial functions of higher degree, with any order of smoothness conditions imposed on the boundary of two d-faces. Besides the applications in curve-fitting, splines found wide applications in the finite element method to estimate solutions of ordinary and partial differential equations. Subsequently, applied mathematicians and engineers working in the areas of curve fitting, finite element methods, computer-aided geometric design, signal processing, mathematical modelling, computer-aided design, computer-aided manufacturing, and circuits and systems started using multivariate splines extensively. In fact, Spline functions are most successful approximating functions for practical purposes till today. In regression models, regression splines have several benefits when compared to linear and polynomial regressions. Unlike polynomials, which must use a high degree polynomial to produce flexible fits, splines introduce flexibility by increasing the number of knots but keep the degree fixed.

By their definition, the study of spline functions involves both algebra and geometry, and the smoothness conditions also require an understanding of analysis. In depth understanding of splines and their applications involved an interplay between the underlying combinatorics, geometry of the subdivision and the algebraic/analytic properties of the resulting set of functions. This interplay became domain of active research with the pioneering works of P. Alfeld , P. Alfeld and L.L Schumaker , L.J Billera , L. J. Billera and L. L. Rose [[17],[18],[93]].

The chronology of the study is as follows

Strang [18] conjectured a formula for the dimension of $C^1_d(\Delta)$ for a d-dimensional complex Δ as a vectorspace , which was proved by Morgan and Scott [83], for $d \geq 5$, by construction of a locally supported basis. Alfeld-Schumaker [5] extended Strang's formula to a formula for $dim_d C^r_d(\Delta)$ for $d \geq 4r + 1$, which was further improved in several studies. Billera [17] introduced techniques from homological and commutative algebra to the study of splines, giving an algebraic approach to computing the dimension of $C^r_d(\Delta)$. Subsequently, this algebraic approach has been refined by a number of authors in their research studies who worked on identifying the dimension and bases for the vector space $C^r_k(P)$ of splines[41]. Gilbert, Polster, and Tymoczko [55] generalized the notion of splines, building them on the dual graphs of the polytopes and it was shown in [18] that the two constructions (on polytopes and their duals) are equivalent. Later they constructed these splines over arbitrary edge labeled graphs G and termed these as generalized splines.

The definition of an edge labeled graph and generalized spline rings as defined in [55] are as follows

- **Definition**[55]
  Let $G = (V, E)$ be a graph. Let R be an arbitrary commutative ring with identity which is also an integral domain and let S denote the set of all non-zero ideals of R. Let a function $\alpha : E \longrightarrow S$ be an edge labeling function defined on G,where α labels each edge in graph G by the ideals of the ring R. Then the graph G with function α is called an edge labeled graph which is denoted by $G = (V, \alpha)$.

  The definition of generalized splines over an arbitrary graph G is as follows

- **Definition** [55]
  Let $G = (V, E)$ be a graph of order n. Let R be a commutative ring and let I denote the set of all ideals of R. Let $\alpha : E \longrightarrow I$ be an edge labeling. A generalized spline of (G, α) is a vertex labeling $F : V \longrightarrow R$ such that for each edge uv, $F(u) - F(v) \in \alpha(uv)$ where $F(u) \in R$ for each vertex in u in V.This condition is known as edge condition or GKM condition satisfied by the generalized splines over the edges of the graph G.The set of splines defined over G is denoted by $R_{(G,\alpha)}$. Each element of $R_{G,\alpha}$ is called a generalized spline. If the edge labeling is clear, it is denoted as R_G.

  The following figures (as discussed in [55]) are two examples of generalized splines R_{C_4} and R_{K_4}, defined on the 4-cycle C_4 and the complete graph K_4.

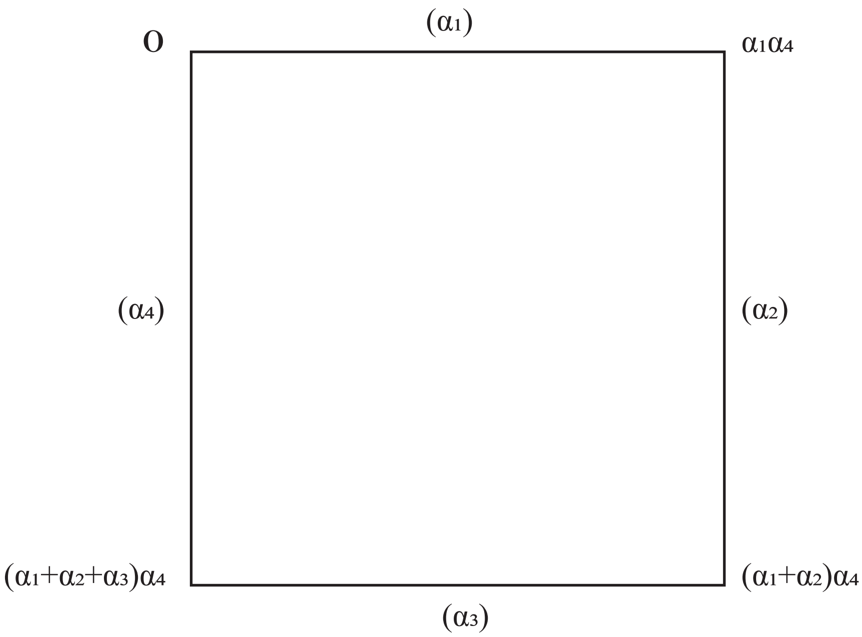


FIG. 7.2: Generalized spline on C_4

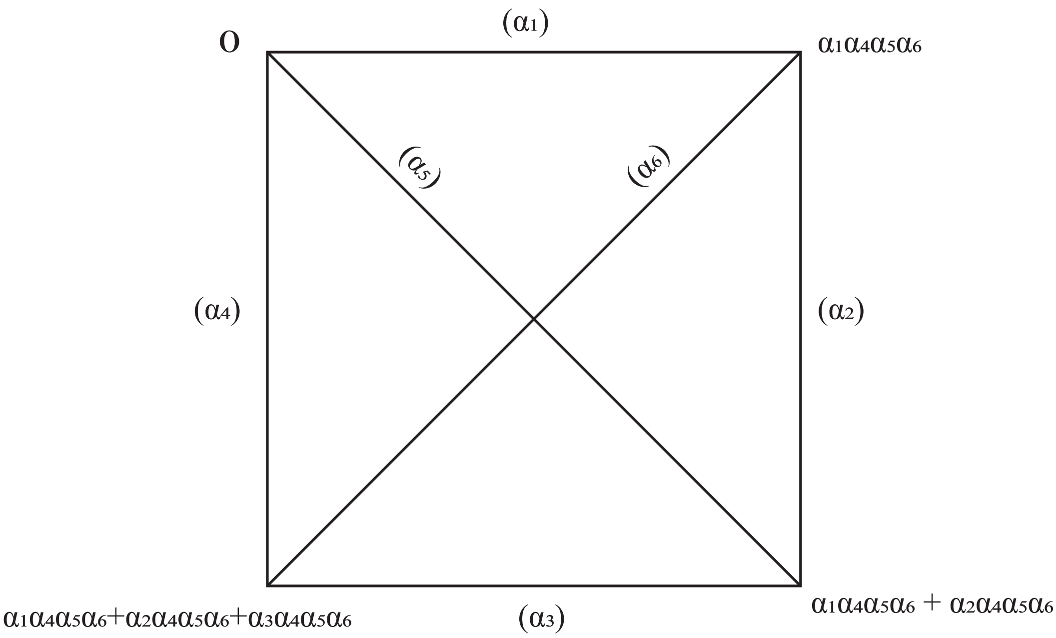


FIG. 7.3: Generalized spline on K_4

Algebraically, the set of splines over a subdivision of domains was seen to be a subring of the product ring $R \times R \times \dots R$ (n copies), where R was the ring of polynomials and n denoted number of subdivisions of the domain. Also, it was observed that the above spline ring was a module over the ring of polynomials[55]. Identifying the dimension and suitable bases for the free spline modules became an active area of research for mathematicians, still remaining far from being completely understood.

As discussed in the previous paragraph, Simcha Gilbert, Shira Polster and Julianna Tymoczko [55], expanded the family of objects on which these splines were defined to arbitrary graphs, which they called the generalized splines denoted by $R_{(G,\alpha)}$. They have shown that $R_{(G,\alpha)}$ has a ring and module structure over the ring R, for an arbitrary graph G [55] and have shown that there exists a minimum generating set which functions like a basis for R_G over an integral domain R. In fact, over an integral domain the spline module contains a free sub-module of rank at least the number of vertices in G and over a PID, the spline module is always free with rank exactly equal to the number of vertices in G. Handschy, Melmick and Reinders have studied the generalized spline modules on cycle graphs over the ring of integers [63]. They have shown the existence of flow-up basis for the generalized spline modules on cycle graphs, thus proving that these spline modules are free. Bowden and Tymoczko [21] considered the module of generalized splines over the quotient ring , $\mathbb{Z}/m\mathbb{Z}$ where m is an integer, which is not an integral domain . It was shown that finite generalized modules are generally not free, but the minimum generating sets function like bases and it was shown in [21] that over the quotient ring $\mathbb{Z}/m\mathbb{Z}$, these minimum generating sets are smaller than expected. As the rank of the Z-module of splines is defined to be the number of elements of a minimum generating set,it was concluded that these modules can have any rank r with $1 \leq r \leq n$.

Nealy Bowden, Sarah Hagen and Stephanie Reinders [20] proved that flow-up classes with smallest leading entries form a module basis for $R_{(G,\alpha)}$, where R is an integral domain. In [50], Gjoni studied integer generalized splines on cycles and gave basis criteria for $Z_{(C_n,\alpha)}$ via determinant of flow-up classes. Emmet Reza Mahdavi [74] characterized integer generalized splines on diamond graph and developed a determinantal criterion for a given set of splines to form a basis. Also in [7], Selma Altinok and Samet Sarioglan proved the existence of flow-up bases for generalized spline modules on arbitrary graphs over principal ideal domains. They introduced a method to determine the smallest leading entries of flow-up classes on arbitrary graph over a principal ideal domain by using zero trails and gave an algorithm to determine flow-up classes on arbitrary ordered cycles. In [8], Selma Altinok and Samet Sarioglan generalized that work and gave basis criteria for diamond graphs and trees over any GCD domain.

7.3 Rationale of the Study

As discussed earlier, Gilbert et.al[55] have shown that the set of splines over an arbitrary graph G has a ring as well as module structure over the base ring R. They completely described the generalized spline modules on trees, while leaving open the investigation on cycles[55]. They have shown that when R is a domain then the rank of the module $R_{(G,\alpha)}$ is equal to $\mid V \mid$, number of vertices in G.

We have extended the study further in our research and addressed the open questions posed by Gilbert, Polster and Tymoczko in [55]. We have constructed nontrivial generalized splines for the special cases of G, where G is a Complete graph, Complete Bipartite graph with any number of vertices and Hypercubes. We have developed a general algorithm to express the ring of generalized splines for hypercubes of any dimension , taking into account the bipartite nature and Hamiltonian property of the graph . Also, Python codes were developed which calculated the elements of the generalized spline ring , for complete graphs and complete bipartite graphs.

Bowden and Tymoczko[21] found an algorithm for writing the generating set which acts as a basis for the generalized spline modules for cycle graphs, taking the base ring as the quotient ring of integers modulo m .We have extended their method to a generating set for the wheel graphs which is viewed as a graph extention to the cycle graph.

Selma Altinok and Samet Sarioglan [8] have given basis criteria for graphs obtained by joining cycles, diamond graphs and trees together along common cut vertices.Based on this result [8], we have obtained the basis criteria for $R_{(G,\alpha)}$ on edge labeled Dutch windmill graph and special cases of Dutch windmill graph such as Friendship graph and Butterfly graph which have common cut vertices with cycle graph C_n and triangles respectively, over any GCD domain, by using determinantal techniques and flow-up bases.

We have also studied basis criteria for generalized splines on some isomorphic graphs over GCD domain and constructed flow-up basis for generalized spline modules on arbitrary ordered rooted trees.

Isomorphism in trees establish structural similarities between them, thereby finding variety of applications in diverse disciplines such as social sciences, science and technology, organisational structures etc. Subtree Isomorphisms are used where pattern matching plays a significant role in searching the identical problems which can be compared. We have shown that isomorphic graphs have the same flow up basis over any GCD domain. Further, we have used the zero trail method to construct the flow up basis for some tree graphs and in general, developed an algorithm for indexing the vertices of an ordered rooted tree graph.This helps us in constructing a flow-up basis for any ordered rooted tree graph and its isomorphic graphs.

Thus we have studied ring and module of generalized splines over a variety of graphs considering base rings which are either GCD domains,integral domains or quotient rings.However we have also realized that our study has generated several areas which can be taken up for further study and finding applications of the algorithms for the real world problems.

7.4 Objectives of the study

- To identify the generalized spline ring $R_{(G,\alpha)}$ for G to be a graph in an important family of graphs such as wheel graphs, complete graphs, complete bipartite graph and hypercubes. All these graphs are extensively used in network theory.
- To develop algorithms and software codes to write down the elements of the generalized spline rings defined over the above mentioned graphs for any number of vertices.
- To show the existence of flow up basis and construct these basis for the generalized spline modules over the graphs which are joins of cycle graphs such as the Dutch Windmill graphs and their special cases such as the butterfly graphs and friendship graphs.
- To find the minimum generating sets over the quotient rings $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/p^k\mathbb{Z}$, where $m = m_1 m_2 \ldots m_r$ and each m_i and p is prime, for the wheel graph, which is an extension to the cycle graph.
- To investigate whether ring automorphism together with graph isomorphism establishes module isomorphism between the modules of generalized splines over isomorphic graphs.
- To investigate whether a set of generalized splines over a graph, satisfying the basis criterion defined over GCD domain is equivalent to the corresponding set of generalized splines over isomorphic graphs. Indexing the vertices of an arbitrary ordered rooted tree graph in a way that an algorithm for writing down the flow up basis satisfying the basis criterion can be obtained.

7.5 Preliminaries

In this section we give the definitions and the results used in our work.

The definition of flow-up classes and constant flow-up splines are as follows:

- **Definition** [8]

  Let (G, α) be an edge labeled graph with $V(G)$ =$\{v_1$, v_2,. . .,$v_n\}$ and $E(G)$ = $\{e_1$, e_2,. . ., $e_m\}$.Let $1 \leq i \leq n$. The set of all generalized splines $F : V \longrightarrow R$ where $F(v_j) = 0$ if $j < i$ and $F(v_i) \neq 0$ is called a flow-up class and is denoted by $F_{(i)}$. Thus a generalized spline in $F_{(i)}$ has $i - 1$ leading zeros and the class $\{F_{(i)}\}$ is a submodule of $R_{(G,\alpha)}$.

- **Definition**[21]

  Given a graph G with an ordered set of vertices $V = (v_1, v_2, \ldots, v_n)$, a flow-up spline for a vertex v_i is a spline f_i for which $(f_i)_{v_i}$= 0, whenever $k < i$.A constant flow-up spline in $\mathbb{Z}/m\mathbb{Z}$ is a flow-up spline p for which there exists an element $n_i \in \mathbb{Z}/m\mathbb{Z}$ such that $p_{i_v} \in \{0, n_i\}$ for each $v \in V$.i.e,The entries of constant flow-up splines have at most one possible nonzero value.

  The definition of zero trail which is given in [7]is as follows

- **Definition** [7]

  Let $G = (V, E)$ be a graph with an edge labeling α. Let $u, v \in V$. A $u - v$ trail in G is an alternating sequence $T = (\ u = v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_k}, v_{i_k} = v)$ of vertices and edges such that $e_{ij} = v_{i(j-1)}v_{ij}$ and all the edges in T are distinct. If $\alpha(e_{ij}) = l_{ij}$, then the trail T is denoted by l_{i_1}, l_{i_2},. . .,l_{i_k}. If $v_{i_k} = 0$, then T is called a zero trail and is denoted by $T^{(u,0)}$. Also gcd and lcm of $\{\ l_{i_1}, l_{i_2}, \ldots, l_{i_k}\}$ are denoted by $(\ T) \ = (\ l_{i_1}, l_{i_2}, \ldots, l_{i_k})$ and $[\ T\ ] = [\ l_{i_1}, l_{i_2}, \ldots, l_{i_k}]$ respectively.

  We discuss the example in [7] to explain the zero trails of a vertex.

- **Example**[7]

  Let (G_1, α) be the edge labeled graph (Fig 7.4) which is given as in [7] and $(0, 0, f_3, f_4, f_5) \in F_3$ where F_3 is the flow-up class as discussed before. The zero trails of v_3 are shown as red and blue lines in Fig.7.4.

  The zero trails of v_3 are listed below $p_1^{(3,0)} = l_7l_4, p_2^{(3,0)} = l_7l_5l_3, p_3^{(3,0)} = l_7l_5l_2, p_4^{(3,0)} = l_6l_3, p_5^{(3,0)} = l_6l_2, p_6^{(3,0)} = l_6l_5l_4$

  The set of all greatest common divisors of zero trails of v_3 is given as $\{(p^{(3,0)})\} = \{(l_7, l_4), (l_7, l_5, l_3), (l_7, l_5, l_2), (l_6, l_3), (l_6, l_2), (l_6, l_5, l_4)\}$.

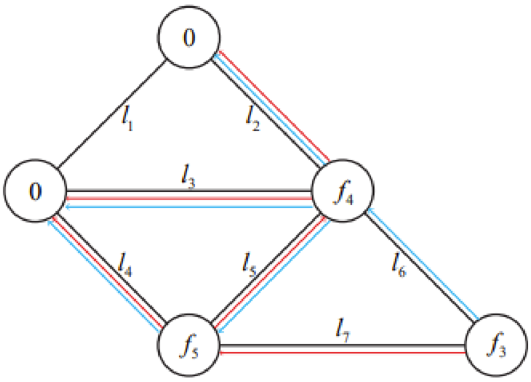


FIG. 7.4: Zero Trails

The following is definition of the matrix representation of set of flow-up splines.

- **Definition** [8]

  Let (G, α) be an edge labeled graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $A = \{F_1, F_2, \ldots, F_k\} \subset R_{(G,\alpha)}$. Let $F_i(\,v_j) \;= f_{ij}$. Then the n $\times$ k matrix

$$\begin{bmatrix} f_{1n} & f_{2n} & \ldots & f_{kn} \\ \vdots & \vdots & \ldots & \vdots \\ f_{11} & f_{21} & \ldots & f_{k1} \end{bmatrix}$$

is called the matrix representation of A.

The following is the formula of Q_G for a graph G, used in finding basis criteria for generalized spline modules on cycle graph and diamond graph etc, which is given in [8].

- **Definition** [8]

  Let (G,α) be an edge labeled graph with k vertices. Fix a vertex v_i on (G,α) with $i \geq 2$. Label all vertices v_j with $j < i$ by zero. By using the notations in [8], we define Q_G as

  $Q_G = \prod_{i=2}^{k}$ [{($p_t^{(i,0)}$) } | t = 1,...,m_i] ,where $p_t^{(i,0)}$ are zero trails of v_i and m_i is the number of the zero trails of v_i.

  We state the theorems proved in [8], which are used for obtaining basis criterion for a set of flow-up classes to become a basis for certain classes of graphs we have studied.

  The following lemma gives formula Q_G for cycle graph C_n.

- **Lemma**[8]

  Let (C_n,α) be an edge labeled n-cycle. Then $Q_{C_n} = \dfrac{l_1 l_2 \ldots l_n}{(l_1, l_2, \ldots, l_n)}$ where l_1,l_2,. . . ,l_n represent the edge labels of cycle graph.

  Since a tree graph is a graph without cycles,the following lemma gives formula Q_G for G to be a tree.

- **Lemma** [8]

  Let G be a tree with n vertices and k edges. Then $Q_G = l_1 \ldots l_k$ where $l_1, \ldots, l_k$ are edge labels of G .

  The following result gives the basis criteria for cycle graph and tree graph with n vertices as calculated in [8].

- **Theorem** [8]

  Let $(C_n$,$\alpha)$ be an edge labeled n-cycle and let $\{F_1, \ldots, F_n\} \subset R_{(C_n,\alpha)}$. Then $\{F_1, \ldots, F_n\}$ forms a basis for $R_{(C_n,\alpha)}$ if and only if $\mid F_1 F_2 \ldots F_n \mid = r \cdot Q_{C_n}$, where $r \in R$ is a unit.

  The following result which is given as corollary to Theorem 3.26 in [8] generalizes the basis criterion for join of cycles, diamond graphs and trees along common cut vertices.

- **Corollary** [8]

  Let $\{G_1, \ldots, G_k\}$ be a collection of cycles, diamond graphs and trees and let G be a graph obtained by joining $G_1, \ldots, G_k$ together along common vertices which are cut vertices in G. Then $\{F_1,...,F_n\} \subset R_{(G,\alpha)}$ forms a basis for $R_{(G,\alpha)}$ if and only if $\mid F_1\ F_2 \ldots F_n \mid = r \cdot\ Q_{G_1} \ldots Q_{G_k}$, where $r \in R$ is a unit.

  We particularly consider the rooted trees because root of a tree represents the "start" of the data. Another important information assigned to the rooted trees is the ordering of the children (from left to right). Such a tree is called an ordered rooted tree.For example, suppose we have a tree with a single parent and two leaf nodes. So we assign "()" to the leaves. When we move towards the parent node, we combine the parentheses of leaves like "()()" and wrap it in another pair of parentheses like "(()())" and assign it to the parent. This process continues iteratively until we reach the root node.We can encode any ordered rooted tree by assigning a string of 0's and 1's, which uniquely determine the tree by using the AHU algorithm. In AHU algorithm, parenthetical tuples are assigned to all tree vertices. However, these parenthetical tuples have no ordering. Replacing "(" with "1" and ")" with "0", the parenthetical names are converted into canonical names, which can be sorted lexicographically.

We have seen that if two graphs are isomorphic then the zero trails of their vertices are identical.As a result we have concluded that isomorphic graphs have equivalent bases and equivalent basis criteria whenever generalized modules on these graphs are free or generating sets exist.The result does not have strong implications for arbitrary graphs over GCD domains as no polynomial-time algorithm exists for checking the isomorphism between graphs in general.however,we have the AHU algorithm for tree graph isomorphisms, our results are generalized over ordered rooted tree graphs which form a very important class of graphs in Computer Network Theory.

We now discuss the AHU algorithm as in [25], for the isomorphism in trees.

- **AHU Algorithm[25]**
  This Algorithm determines tree isomorphism in time $O(|V|)$ by associating a tuple with each vertex of a tree that describes the complete history of its descendants.

  The AHU [25] algorithm is a serialization technique for representing the vertices of a tree as unique string and is able to capture a complete history of a tree's degree spectrum and structure, ensuring a deterministic method of checking tree isomorphisms.In this algorithm, leaf nodes are assigned with a parenthesis "()". Every time we move upwards, we combine, sort and wrap the parentheses.We can't process a node until we have processed all its children.

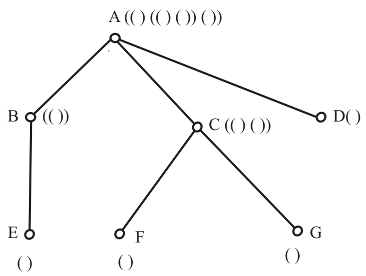


FIG. 7.5: Encoded ordered rooted tree using the AHU Algorithm

Next we give the definition of isomorphism in ordered rooted trees.

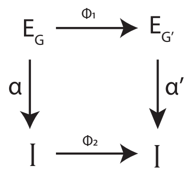
- **Definition[1]**
  Two ordered rooted trees are isomorphic if there exists an isomorphism of rooted trees, such that it preserves the order of children of every vertex.

  Gilbert [55] has shown that the set of generalized splines $R_{(G,\alpha)}$ has a ring as well as a module structure over the base ring R. Also, the following definition and proposition given by him shows that isomorphism in graphs together with ring automorphism induces isomorphism between the ring of generalized splines.

  Isomorphism of edge labeled graph is as follows

- **Definition** [55]

  Let (G, α) and $(G^{'}, \alpha')$ be edge labeled graphs and R be a commutative ring. A homomorphism of edge labeled graphs $\phi : (G, \alpha) \longrightarrow (G^{'}, \alpha^{'})$ is a graph homomorphism $\phi_1 : G \longrightarrow G^{'}$ together with a ring automorphism $\phi_2 : R \longrightarrow R$, so that for each edge $e \in E_G$, we have $\phi_2(\alpha(e)) = \alpha^{'}(\phi_1(e))$ i.e,the following diagram is commutative.



  An isomorphism of edge–labeled graphs is a homomorphism of edge–labeled graphs whose underlying graph homomorphism is an isomorphism.

- **Proposition**[55]

  If $\phi : (G, \alpha) \longrightarrow (G^{'}, \alpha^{'})$ is an isomorphism of edge–labeled graphs then ϕ induces an isomorphism of the corresponding rings of generalized splines $\phi_* : R_G \longrightarrow R_{G'}$ defined as $\phi_*(p)_{\phi_1(u)} = \phi_2(p_u)$ for each u $\in V_G$. Using the above proposition, we have shown in section 5, that isomorphic graphs together with ring automorphism also induce module isomorphism ϕ^* between $R_{(G,\alpha)} \cong R_{(G',\alpha')}$.

  With these preliminaries, we discuss the results we have obtained in this study.

7.6 Results and Discussions

We discuss the four major results of this research study starting with extension of work done by Nealy Bowden and Julianna Tymoczko on cycles [21] to classify splines on wheel graphs.

- **Splines on Wheel Graphs over the quotient ring $\mathbb{Z}/p^k\mathbb{Z}$**

  Referring to the results given by Nealy Bowden and Julianna Tymoczko [21],we applied the study to wheel graphs which are extension to cycle graphs. We have proved the following theorem which gives a generating set in matrix form for the generalized spline module $R_{W_{n+1}}$ over quotient ring $\mathbb{Z}/m\mathbb{Z}$.

  Considering the Wheel graph W_{n+1} whose edges are labeled by some powers of $a \in R$,we have

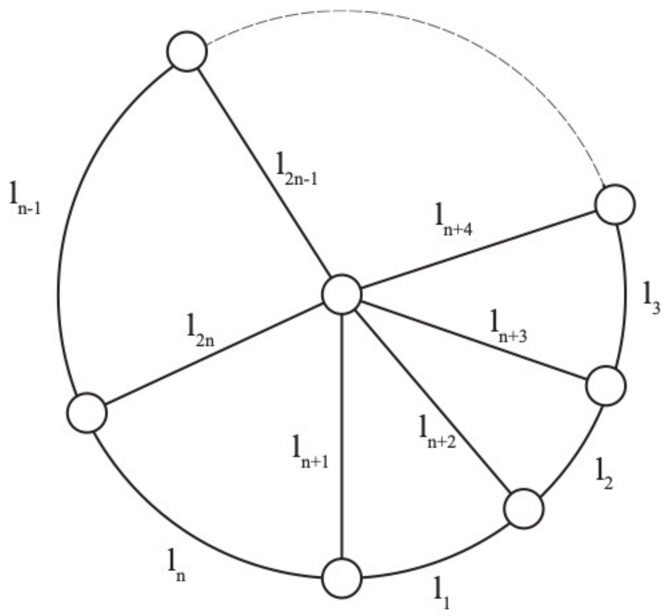


FIG. 7.6: Wheel Graph

- **Theorem**

  Let a be a zero divisor in $\mathbb{Z}/m\mathbb{Z}$. Suppose all of the edges of W_{n+1} are labeled with powers of a and the set of edge labels is $(a^{k_1}, a^{k_2}, \ldots, a^{k_n})$. Without loss of generality assume that a^{k_1} is the minimal power in the set and that a^{k_1} is the label on the edges $l_n, l_{n+1}, l_{n+2}, \ldots, l_{2n}$. Then the following matrix contains all generating $n+1$ splines on W_{n+1}.

$$B(R_{w_{n+1}}) = \begin{bmatrix} 1 & l_1 & \ldots & l_i & \ldots & l_{n-1} & l_n & l_n \\ 1 & l_1 & \ldots & l_i & \ldots & l_{n-1} & l_n & 0 \\ 1 & l_1 & \ldots & l_i & \ldots & l_{n-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 1 & l_1 & \ldots & 0 & \ldots & \vdots & \vdots & 0 \\ 1 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

  Here l_i is equal to the edge label a^{k_j} where $1 \leq j \leq n-1$. As a corollary to the above result,we have

- **Corollary**

  Let W_{n+1} be a wheel graph with $n+1$ vertices, p be a prime number and k be a positive integer.Then the splines on $R_{W_{n+1}}$ over $\mathbb{Z}/p^k\mathbb{Z}$ are generated by the minimum generating set B in the above result.

  Now we give following theorem which gives a flow-up generating set for the generalized splines on C_n over the ring $\mathbb{Z}/m\mathbb{Z}$.

- **Theorem**
  Let C_n be a cycle and R be ring $\mathbb{Z}/m\mathbb{Z}$ where $m = m_1\ m_2$ and m_1,m_2 are prime factors of m.Let the edges of C_n be labeled by prime factors of m such that each prime factor of m appears at least once in the edge labeling of C_n. Then the following set is a flow-up generating set for R_{C_n}.

$$B(R_{C_n}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_{n-1}l_n \\ l_{n-2}l_{n-1} \\ \vdots \\ l_2l_3 \\ l_1l_2 \\ 0 \end{pmatrix} \begin{pmatrix} l_{n-1}l_n \\ \vdots \\ l_3l_4 \\ l_2l_3 \\ 0 \\ 0 \end{pmatrix} \dots \begin{pmatrix} l_{n-1}l_n \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

- **Remark**
  When m has only two prime factors the generating set B may not be minimum. It will lose a rank whenever two adjacent edges of C_n are labeled with distinct primes m_1 and m_2.We observe that when m has more than 2 prime factors, the generating set doesn't loose rank since every vertex in cycles is incident to only two edges.

  We also give following theorem which gives generating set for the splines on W_{n+1} over the base ring $\mathbb{Z}/m\mathbb{Z}$.

- **Theorem**
  Let W_{n+1} be a cycle and R be a ring $\mathbb{Z}/m\mathbb{Z}$ where $m = m_1\ m_2$ and m_1,m_2 are primes. Let the edges of W_{n+1} be labeled by either by m_1 or m_2 such that both m_1 and m_2 appear as edge labels atleast once. Then the following set is a generating set for $R_{W_{n+1}}$ over the base ring $\mathbb{Z}/m\mathbb{Z}$.

$$B(R_{W_{n+1}}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}\dots l_{2n} \\ l_{n-2}l_nl_{2n} \\ \vdots \\ l_2l_3l_{n+3} \\ l_1l_2l_{n+2} \\ 0 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}\dots l_{2n} \\ l_{n-2}l_nl_{2n} \\ \vdots \\ l_2l_3l_{n+3} \\ 0 \\ 0 \end{pmatrix} \dots \begin{pmatrix} l_{n+1}l_{n+2}\dots l_{2n} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

  The following theorem completely characterise the situation, when the generating set of the module $R_{W_{n+1}}$ will be minimum.

- **Theorem**
  Let W_{n+1} be a wheel graph, with vertices $v_1, v_2, \dots, v_{n+1}$ and edges $l_1, l_2, \dots, l_{2n}$

and $m = m_1 m_2 \ldots m_r$ where $m_1, m_2, \ldots, m_r$ are primes. Let each edge of the wheel graph be labeled by the prime factors of m. Then, we can get the generating set B of $R_{W_{n+1}}$ consisting of flow-up splines by labeling the vertices of W_{n+1} with the product of edge labels of edges incident on them . The above set will be minimum whenever the number of prime factors of m is greater than n, i.e, the number of vertices in the cycle graph C_n.

The following is second major result of our research study.

- **An Algorithm for Generating Generalized Splines on graphs such as Complete Graphs,Complete Bipartite Graphs and Hypercubes** In this section of our work, we have extended the work done by Simcha Gilbert, Shira Polster, and Julianna Tymoczko [55] to identify and construct ring of generalized splines $R_{(G,\alpha)}$ for some important family of graphs like complete graphs, complete bipartite graphs and Hypercubes.We have found algorithms for generating generalized spline rings on these graphs.

- **Generalized splines for Complete graphs,**$K_n, n \geq 3$
We have used Theorem 3.8 from [55] to identify the ring R_G, for the complete graph K_n, for $n \geq 3$.

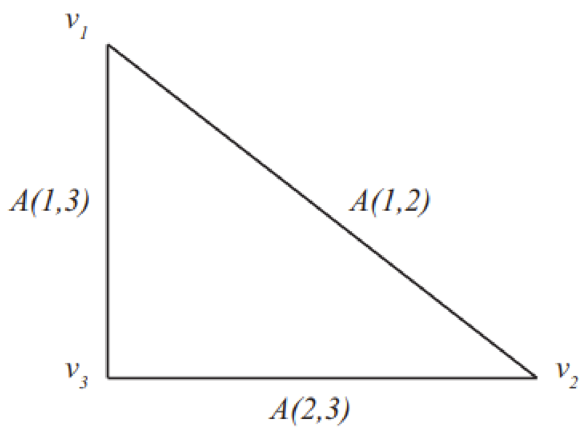


Fig. 7.7: Generalized spline on K_3

First we discuss nontrivial generalized splines for complete graph K_3 as given in [55]. Here the edges (v_1, v_2), (v_2, v_3) and (v_3, v_1) of the graph K_3 are labeled with the non-zero ideals $A(1, 2)$, $A(2, 3)$ and $A(3, 1)$ respectively of the ring R, when R is an integral domain.

It follows from Theorem 3.8 in [55], a generalized spline p_{K_3} on the complete graph K_3 is

$$p_{K_3} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3) \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3) \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \end{bmatrix}$$

Here $\alpha(i,j)$ represents generator of the edge ideal $A(i,j)$. It is seen that p_{K_3} satisfies the edge conditions on K_3, because if the vertices v_i and v_j are adjacent, then $p_{v_i} - p_{v_j} \in A(i,j)$,as $\alpha(i,j)$ is a factor of $p_{v_i} - p_{v_j}$

Since R is an integral domain and each $\alpha(i,j)$ is not equal to zero, R_{K_3} contains nontrivial generalized splines. Using the above result, we have generated the algorithm for developing the generalized spline for the complete graph K_n, for any $n \geq 4$. As discussed in rationale of the study we addressed the open questions given in [55] to construct generalized splines R_{K_n}, for any $n \geq 4$, each time adding one vertex and joining the new vertex to the remaining $(n-1)$ vertices.

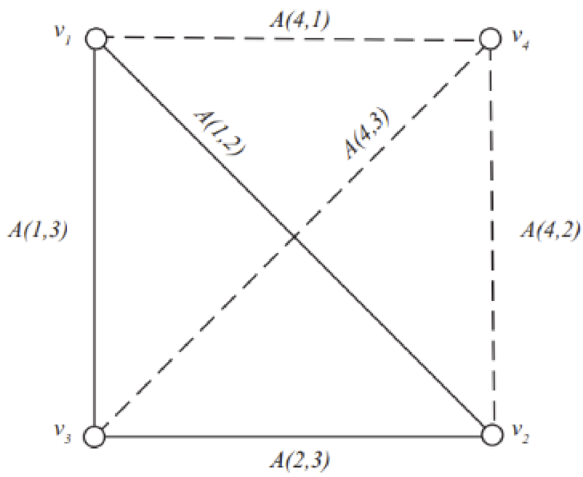


FIG. 7.8: Generalized Spline on K_4

We add the vertex v_4 to K_3 and join the new vertex v_4 with the vertices v_1,v_2,v_3 of K_3 . The new edges are labeled with the non-zero ideals $A(4,1)$, $A(4,2)$, $A(4,3)$ of integral domain R and $\alpha(4,1)$, $\alpha(4,\ 2)$, $\alpha(4,\ 3)$ are the generators of the respective edge ideals. It can be seen that every vertex label for $p_{K_3} \in R_{K_3}$ is multiplied by the factor $\alpha(4,\ 1)\alpha(4,\ 2)\alpha(4,\ 3)$ to get the corresponding vertex labels for the spline $p_{K_4} \in R_{K_4}$, where R_{K_4} denotes the ring of all generalised splines for the edge labeled graph $(K_4,\ \alpha)$. It is easily verified that if the new vertex v_4 is labeled with $p_{v_4} = \alpha(4,\ 1)\alpha(4,\ 2)\alpha(4,\ 3)$, then p_{K_4} becomes a generalized spline for R_{K_4} since the edge conditions are satisfied for the adjacent vertices in K_4. So we have

$$p_{K_4} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \\ \langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \end{bmatrix}$$

$p_{v_1} - p_{v_2} \in A(1,2)$, since $\alpha(1,2) \in A(1,2)$ is a factor of $p_{v_1} - p_{v_2}$. Similarly the edge conditions will be satisfied for other edges also.

Here $p_{v_4} = \alpha(4,1)\alpha(4,2)\alpha(4,3)$ is non-zero because R is an integral domain. Also since K_3 is a sub-graph of K_4 and R_{K_3} contains nontrivial generalized splines "R_{K_4}" also contains nontrivial generalized splines.

We give the following algorithm for writing the generalized spline for complete graph K_n , for any $n \geq 4$.

- **Theorem**

We obtain the complete graph K_n by adding the n^{th} vertex v_n and the edges $(v_n, v_1), (v_n, v_2), \ldots, (v_n, v_{n-1})$ to the complete graph K_{n-1}, labeling the new edges with the ideals $A(n,1), A(n,2), \ldots, A(n, n-1)$ in the ring R, which are generated by the elements $\alpha(n,1), \alpha(n,2), \ldots, \alpha(n, n-1)$.Then the generalized spline ring R_{K_n} contains the elements of the type

$$p_{K_n} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle N_4\rangle \ldots \langle N_n\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle N_4\rangle \ldots \langle N_n\rangle \\ \langle N_4\rangle \ldots \langle N_n\rangle \\ \langle N_5\rangle \ldots \langle N_n\rangle \\ \vdots \\ \vdots \\ \vdots \\ \langle N_n\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \\ p_{v_5} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_n} \end{bmatrix}$$

Here, the notations $N_4, N_5, \ldots, N_n$ are as follows

$N_4 = \alpha(4,1)\alpha(4,2)\alpha(4,3)$

$N_5 = \alpha(5,1)\alpha(5,2)\alpha(5,3)\alpha(5,4)$

$\vdots$

$\vdots$

$\vdots$

$N_n = \alpha(n,1)\alpha(n,2)\ldots\alpha(n, n-1)$

Here $p_{v_1}, p_{v_2}, \ldots$ denote the vertex labels of vertices $v_1, v_2, \ldots, v_n$ of the graph K_n.

We give software code for the above algorithm using Python software. Using this we can obtain generalized spline p_{K_n} for K_n , for $n \geq 3$. In this code we have used $A(i, j)$ as the notation for the ideal as well as for the element of the ideal.

- **Python code for generalized spline modules on Complete graph** K_n The Python code is given as

```

import numpy as np

K3 = np.array(['0',"A{1,2}*A{1,3}","(A{1,2}+A{2,3}*(A{1,3})"])

def generate Kn(n):
if n leq 3 :
return K3

    else:
        ans = K3

        for i in range (4,n+1):

            j= np.hstack([ans," "])

            symbol_arr = list()

            a = " "

            for k in range (1,i):

                a = a +"A{"+str(i)+","+str(k)+"}"

            ans = [ ]

            for x in j:

                if x! = '0':

                    ans.append(x+'*'+a)
                    else:

                    ans.append(x)

        return ans

generate_Kn ( )
```

Listing 7.1: Python code for generalized spline module on Complete graph K_n

We have extended the method to develop an algorithm for writing the elements of the generalized spline ring $R_{K_{n_1,n_2}}$ for the complete bipartite graph K_{n_1,n_2}.We considered the general case of complete bipartite graph , where the vertex sets V_1 and V_2 contain n_1 and n_2 vertices respectively.

- **Theorem**

  Let K_{n_1,n_2} be a complete bipartite graph with vertices partitioned into two disjoint sets V_1 and V_2, consisting of n_1 and n_2 vertices respectively.Then, ordering the vertices in clockwise sense and using the same notations as before, the following $p_{K_{n_1,n_2}}$ gives a generalized spline for the complete bipartite graph K_{n_1,n_2}.

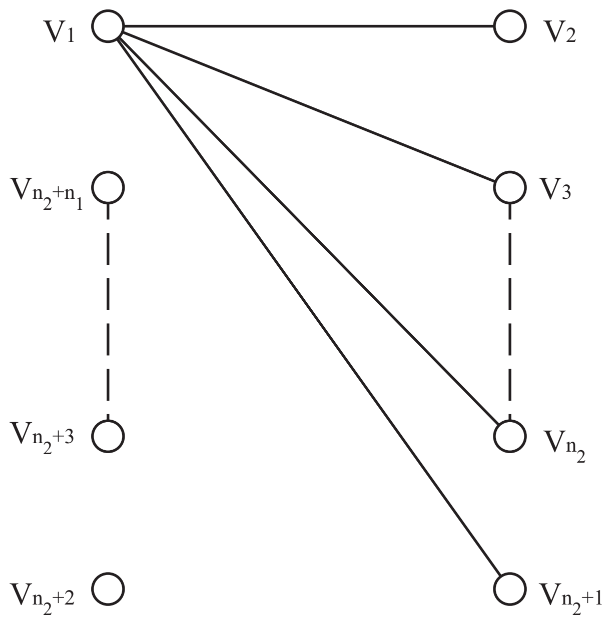


FIG. 7.9: Generalized Spline on Complete Bipartite Graph K_{n_1,n_2}.

$$p_{K_{n_1,n_2}} = \begin{bmatrix} 0 \\ \alpha(1,2)\langle N_{(n_2+2)}\rangle\langle N_{(n_2+3)}\rangle \dots \langle N_{(n_2+n_1)}\rangle \\ \alpha(1,3)\langle N_{(n_2+2)}\rangle\langle N_{(n_2+3)}\rangle \dots \langle N_{(n_2+n_1)}\rangle \\ \vdots \\ \vdots \\ \alpha(1,n_2+1)\langle N_{(n_2+2)}\rangle\langle N_{(n_2+3)}\rangle \dots \langle N_{(n_2+n_1)}\rangle \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ \vdots \\ \vdots \\ p_{v_{n_2+1}} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_{n_2+n_1}} \end{bmatrix}$$

where $N_{n_2+i} = \alpha(n_2+i,2)\ \alpha(n_2+i,3)\ \dots\ \alpha(n_2+i,n_2+1)$,for i = 2,3,... ,n_1. We have also developed the software code for the generalized spline modules on complete bipartite graph using python software which generates the generalized spline $p_{K_{n_1,n_2}}$ for K_{n_1,n_2}, for any value of n_1, n_2.

Next, we give an algorithm for writing the generalized spline for the edge labeled n-dimensional hypercube Q_n, for any n.

Before constructing the generalized splines for the n-dimensional hypercube Q_n, we discuss about the Gray code , which was given by Frank Gray in 1947 to prevent the spurious output from electro-chemical switches. In the present time, they are widely used for error correction in digital communications. The Gray code is an n-bit code which is an ordering of the 2^n strings of length n over 0, 1, such that every pair of successive strings differ in exactly one position. For example a 2-bit Gray code is 00, 01, 11, 10 and a 3-bit Gray code is 000, 001, 101, 111, 011, 010,110, 100. These Gray codes exists for all n [35]. Here we discuss about the n-dimensional hypercube Q_n, which is a regular graph with 2^n vertices, where each vertex corresponds to a binary string of length n [35] . Two vertices labeled by strings x and y are joined by an edge if x can be obtained from y by changing a single bit. The hypercube for n= 1,2,3,4 are shown in the following figures.

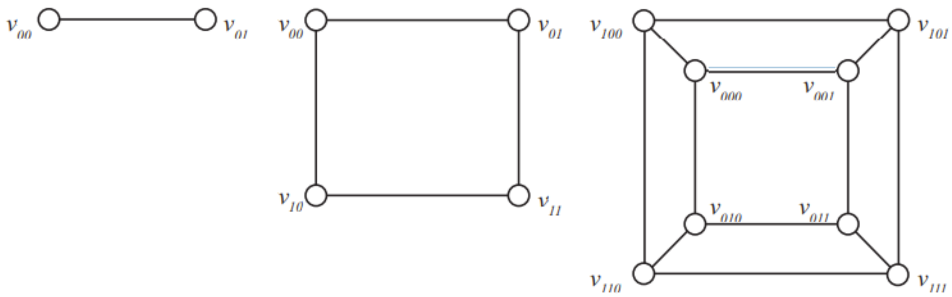


FIG. 7.10: Hypercubes Q_1, Q_2 and Q_3

Interestingly, the existence of one dimensional Gray code is related to a basic property of the n-dimensional hypercube Q_n, which says that for every integer $n \geq 2$, Q_n has a Hamiltonian cycle. Here, the term Hamiltonian cycle means a cycle in a graph G that contains all the vertices exactly once in G [49]. The following figures expresses the Hamiltonian property and bipartite structure of Q_2, Q_3 and Q_4.

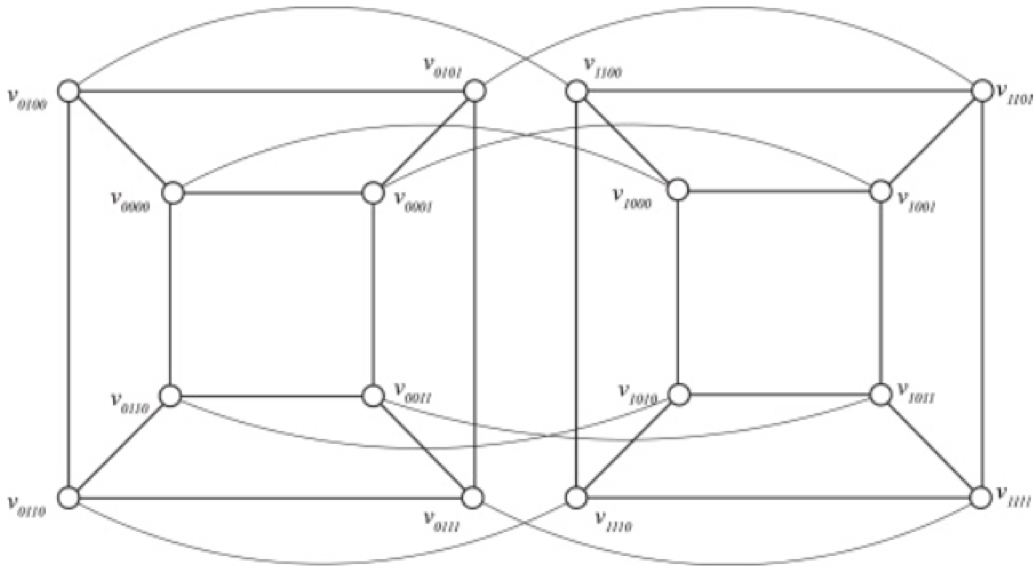


FIG. 7.11: The graph of hypercube Q_4

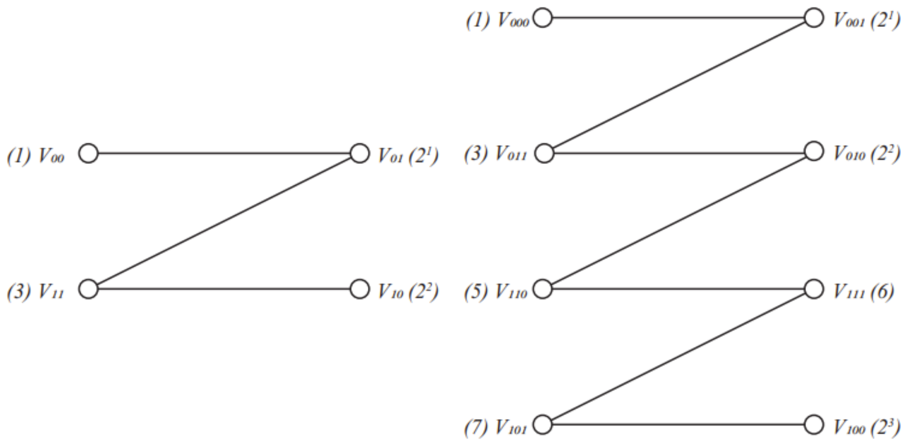


FIG. 7.12: Hamiltonicity of Hypercubes Q_2 and Q_3

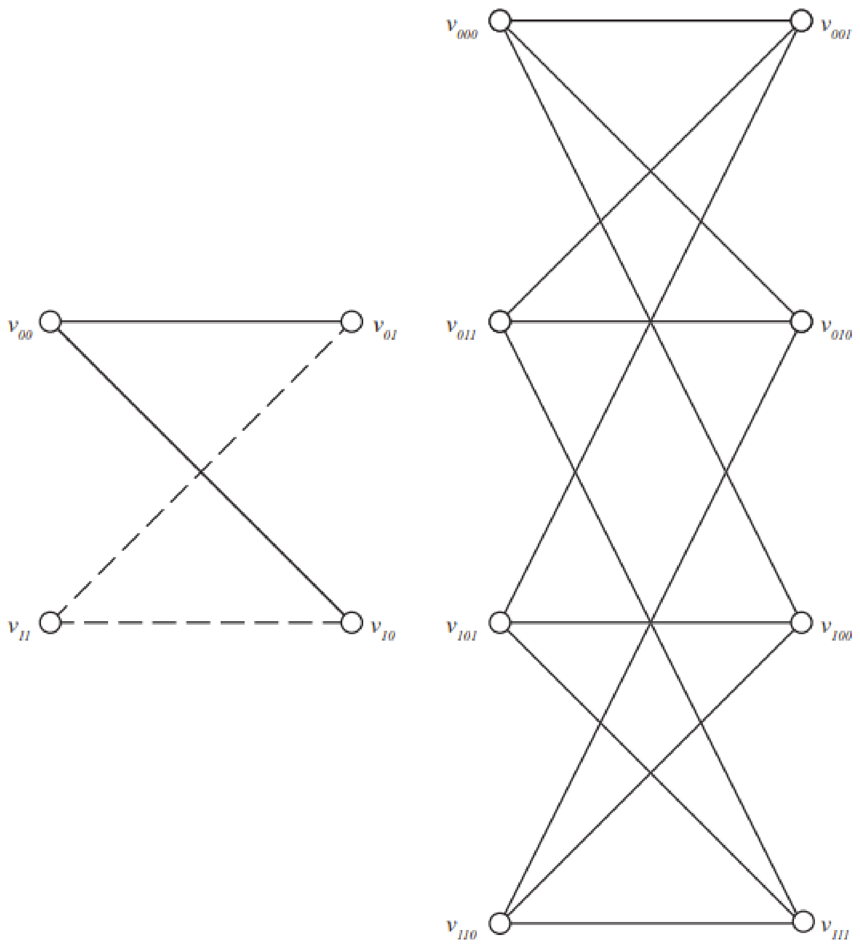


FIG. 7.13: Bipartite structure of Hypercubes Q_2 and Q_3

We define an ordering of the vertices of the hypercube in the same way as they appear in the Hamiltonian cycle. Thus, we number the vertices $1, 2, 3, \dots, 2^n$, with the vertices 2,4,8,... expressed as 2, $2^2, 2^3, \dots, 2^n$ and call this the Hamiltonian ordering. This helps us in identifying pattern in which the non-zero vertex labels appear in the generalized spline for the n-dimensional hypercube. Also, hypercubes are regular graphs with degree of each vertex equal to n. Another important property of hypercubes which we have used in the construction of generalized splines is the bipartite nature of these graphs[35]. This means that the vertex set of hypercube can be partitioned into two subsets V_1 and V_2 such that

1. No vertices of either of the subsets V_1 and V_2 are adjacent to vertices within the same set.

2. Every vertex in V_1 is adjacent to exactly n vertices V_2 and vice versa.

We constructed generalized spline for the graph Q_2 over R which is a commutative ring with identity and also an integral domain. The edges of Q_2 are labeled with

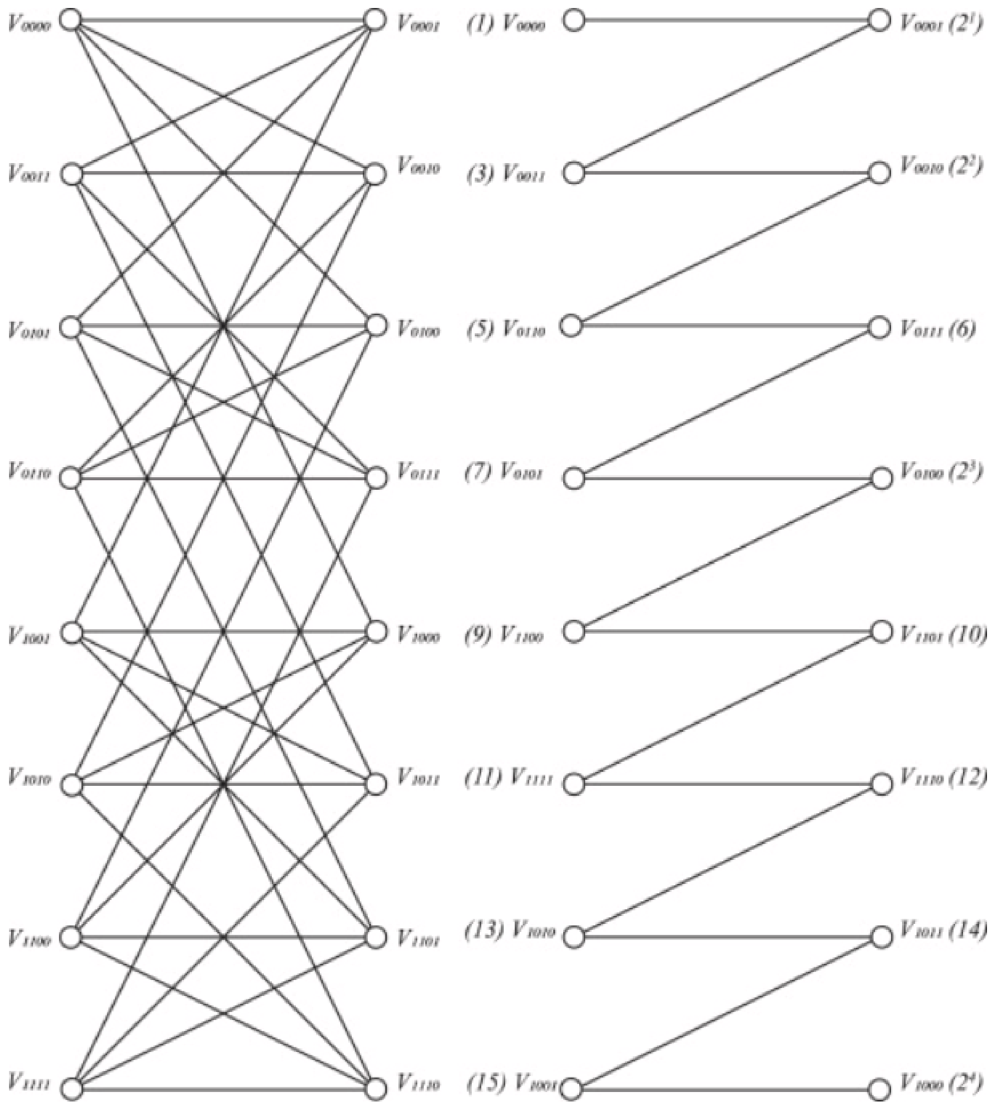


FIG. 7.14: The bipartite structure and Hamiltonian path of the hypercube Q_4

non-zero ideals of R. The vertices are ordered in the way they appear in Hamiltonian cycle.

Then it can be easily verified that a generalized spline for Q_2 is given by:

$$p_{Q_2} = \begin{bmatrix} 0 \\ \alpha_{01,00}\alpha_{01,11} \\ 0 \\ \alpha_{10,00}\alpha_{10,11} \end{bmatrix} = \begin{bmatrix} p_{v_{00}} \\ p_{v_{01}} \\ p_{v_{11}} \\ p_{v_{10}} \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_{2^2}} \end{bmatrix}$$

Here we have used similar notations as in the previous sections, i.e.,$\alpha_{ij,rs}$, (for $i, j, r, s = 0$ or 1) denote a generator of the edge ideal associated with the edge joining the vertices v_{ij} and v_{rs}. Interestingly, we note that the non-zero vertex labels in p_{Q_2} appear for the vertices 2 and 2^2. Next, we construct the generalized spline for Q_3.

To construct the generalized splines for the hypercube Q_3,we refer to the bipartite structure and Hamiltonian ordering of Q_3. Then it can be easily verified that a generalized spline for Q_3 is given by:

$$p_{Q_3} = \begin{bmatrix} 0 \\ \alpha_{001,000}\alpha_{001,011}\alpha_{001,101} \\ 0 \\ \alpha_{010,000}\alpha_{010,011}\alpha_{010,110} \\ 0 \\ 0 \\ 0 \\ \alpha_{100,000}\alpha_{100,011}\alpha_{100,110} \end{bmatrix} = \begin{bmatrix} p_{v_{000}} \\ p_{v_{001}} \\ p_{v_{011}} \\ p_{v_{010}} \\ p_{v_{111}} \\ p_{v_{110}} \\ p_{v_{101}} \\ p_{v_{100}} \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_{2^2}} \\ p_{v_5} \\ p_{v_6} \\ p_{v_7} \\ p_{v_{2^3}} \end{bmatrix}$$

The vertices of Q_3 are $v_{i_1 i_2 i_3}$ where (i_1,i_2,i_3) is a binary string of length 3 and two vertices are adjacent if their respective strings differ only at one position. Also, we see that,the Hamiltonian cycle in Q_3 is one in which the vertices follow a 3-bit gray code 000,001, 011, 010, 110, 111, 101, 100. We again give the Hamiltonian ordering to the vertices in Q_3 by numbering the vertices 000,...,100 as 1,2,...,8. Constructing the generalized spline for Q_3 starts with labeling the vertex v_{000} as 0. Now, the vertices adjacent to v_{000} are v_{100}, v_{010} and v_{001} ,which are numbered as $2,2^2,2^3$ according to Hamiltonian ordering of the vertices. We see that these are the only vertices which are labeled with non-zero elements in p_{Q_3}. Also, the vertex labels of these vertices are obtained by taking the product of the elements belonging to the edge ideals corresponding to the three edges which are adjacent to these vertices. It can be verified that with these vertex labelings, p_{Q_3} becomes a generalized spline for the hypercube Q_3, because the edge conditions are satisfied by the vertex labels of adjacent vertices. We can extend the above method of writing the generalized spline to higher dimensional hypercubes.

For Q_4 we have the first vertex as v_{0000} which is adjacent to the vertices v_{0001},v_{0010} ,v_{0100} and v_{1000}. Using the bipartite structure of Q_4 and Hamiltonian ordering, we get the generalized spline for Q_4 as follows

$$p_{Q_4} = \begin{bmatrix} 0 \\ \alpha_{0001,0000}\alpha_{0001,1001}\alpha_{0001,1010}\alpha_{0100,1100} \\ 0 \\ \alpha_{0010,0000}\alpha_{0010,1001}\alpha_{0010,1010}\alpha_{0100,1100} \\ 0 \\ 0 \\ 0 \\ \alpha_{0100,0000}\alpha_{0100,0011}\alpha_{0100,1010}\alpha_{0100,1100} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_{1000,0000}\alpha_{1000,1001}\alpha_{1000,1010}\alpha_{1000,1100} \end{bmatrix} = \begin{bmatrix} p_{v_{0000}} \\ p_{v_{0001}} \\ p_{v_{0011}} \\ p_{v_{0010}} \\ p_{v_{0111}} \\ p_{v_{0110}} \\ p_{v_{0101}} \\ p_{v_{0100}} \\ p_{v_{1100}} \\ p_{v_{1101}} \\ p_{v_{1111}} \\ p_{v_{1110}} \\ p_{v_{1010}} \\ p_{v_{1011}} \\ p_{v_{1001}} \\ p_{v_{1000}} \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_{2^2}} \\ p_{v_5} \\ p_{v_6} \\ p_{v_7} \\ p_{v_{2^3}} \\ p_{v_9} \\ p_{v_{10}} \\ p_{v_{11}} \\ p_{v_{12}} \\ p_{v_{13}} \\ p_{v_{14}} \\ p_{v_{15}} \\ p_{v_{2^4}} \end{bmatrix}$$

We give an algorithm for writing the generalized spline for the edge labeled n-dimensional hypercube Q_n, for any n.

- **Theorem**

Let Q_n be an n-regular hypercube with the vertices partitioned into two disjoint subsets V_1 and V_2, containing 2^{n-1} vertices each. We introduce the Hamiltonian ordering for the vertices of Q_n so that the vertices are numbered as 1, 2, 3, $2^2, \ldots, 2^n$. Let the first vertex be $v_{00\ldots0}$ in V_1 and adjacent vertices $v_{0\ldots01}$, $v_{0\ldots010}$, $v_{0\ldots100}, \ldots v_{10\ldots0}$ in V_2 which are numbered as 2,2^2, $2^3, \ldots, 2^n$. The vertex labels corresponding to the generalized spline p_{Q_n} defined for Q_n are as follows

1. The vertex $v_{00\ldots0}$ is labeled with the element $0 \in R$, i.e, $p_{v_{0\ldots0}} = 0$.

2. The vertex $v_{0\ldots01}$ which is adjacent to $v_{0\ldots0}$ is labeled as $p_{v_{0\ldots01}}$ and is equal to the product of the n elements belonging to the edge ideals associated with the n edges adjacent to $v_{00\ldots01}$.

Thus, $p_{v_{00\ldots01}} = \alpha_{0\ldots01,0\ldots00}\alpha_{00\ldots01,0\ldots011}\alpha_{00\ldots01,0\ldots0101}\cdots\alpha_{00\ldots01,10\ldots01}$

Similarly the vertex $v_{00\ldots10}$ is labeled as $p_{v_{0\ldots10}}$ which is given as

$p_{v_{00\ldots010}} = \alpha_{0\ldots10,0\ldots00}\alpha_{00\ldots10,0\ldots011}\alpha_{00\ldots10,0\ldots0110}\cdots\alpha_{00\ldots10,10\ldots010}$ and so on.

These are the only vertices with non-zero vertex labels, where each vertex label is a product of n elements belonging to n edge ideals and the remaining vertices are labeled as zero. It can be easily verified that p_{Q_n} is a generalized spline on the hypercube Q_n as the edge conditions are satisfied for the adjacent vertices and also, p_{Q_n} is nontrivial since R is an integral domain.

The following is third major result of our research study.

- **Module Basis for Generalized Spline Modules**

  In this section, we have determined conditions for a subset of $R_{(G,\alpha)}$ to form a basis when G is a Dutch windmill graph and Complete graph K_4, when R is GCD domain. We have given basis criteria for $R_{(G,\alpha)}$ on edge labeled Dutch windmill graph and special cases of Dutch windmill graph such as Friendship graph and Butterfly graph which have common cut vertices with Cycle graph C_n and triangles respectively, over any GCD domain.

  The definition of Q_G[8],as discussed in section 4.5, gives a necessary and sufficient condition for the existence of basis for the generalized spline modules over the cycle graph C_n and tree graph as in [8].

  We can obtain the basis criteria for generalized spline modules on Butterfly graph $D_3^{(2)}$ and Friendship graph $D_3^{(m)}$, which are special cases of Dutch windmill graph $D_n^{(m)}$ over any GCD domain.

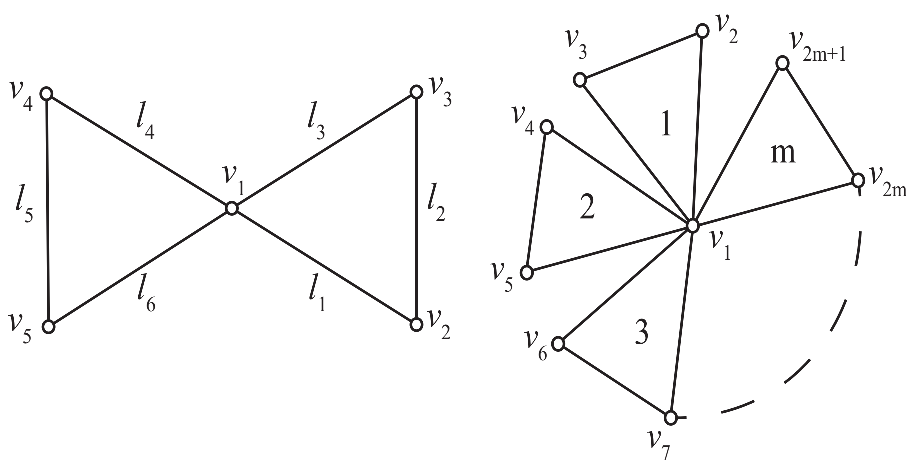


FIG. 7.15: (a) Butterfly Graph $D_3^{(2)}$ (b) Friendship Graph $D_3^{(m)}$

An edge labeled Butterfly graph has 5 vertices v_1, v_2, v_3, v_4, v_5 and 6 edges l_1, l_2, l_3, l_4, l_5, l_6. Using Corollary 3.27[8], we know that flow-up basis for butterfly graph over any GCD domain exists, as it has common cut vertex between two triangles T_1 and T_2 (cycle graphs with 3 vertices). From the definition of Q_G[8], for any graph

G, we have $Q_{D_3^{(2)}}$

$$\begin{aligned} Q_{D_3^{(2)}} &= [l_1,(l_2,l_3)].[l_2,l_3].[l_4,(l_5,l_6)].[l_5,l_6] \\ &= \frac{l_1(l_2,l_3)}{(l_1,(l_2,l_3))}.\frac{l_2l_3}{(l_2,l_3)}.\frac{l_4(l_5,l_6)}{(l_4,(l_5,l_6))}.\frac{l_5l_6}{(l_5,l_6)} \\ &= \frac{l_1l_2l_3}{(l_1,l_2,l_3)}.\frac{l_4l_5l_6}{(l_4,l_5,l_6)} = Q_{T_1}.Q_{T_2} \end{aligned}$$

Next we give condition for basis criterion for $D_3^{(2)}$.

- **Theorem:**
  Let ($D_3^{(2)}$,α) be an edge labeled Butterfly graph over any GCD domain R. Then,

  (i) Dimension of ($D_3^{(2)}$,α) = 5.
  (ii) If F = $\{F_1, F_2, F_3, F_4, F_5\} \subset R_{(D_3^{(2)},\alpha)}$ and the determinant of the matrix of set F is equal to $\mid F \mid = \mid F_1F_2F_3F_4F_5 \mid$, then F is a basis for $R_{(D_3^{(2)},\alpha)}$ if and only if $\mid F \mid = Q_{D_3^{(2)}} = r.Q_{T_1}Q_{T_2}$ where r $\in$ R is a unit.

  In the next lemma, we apply the above result for Friendship graph $D_3^{(m)}$.

- **Lemma**
  Let $(D_3^{(m)}, \alpha)$ be an edge labeled Friendship graph $D_3^{(m)}$ with 2m + 1 vertices $v_1, v_2, \ldots, v_{2m+1}$ and 3m edge labels $l_1, \ldots, l_{3m}$. It is obtained by joining m copies of triangles, $T_1, T_2, \ldots, T_m$ together along the common vertex v_1, which is cut vertex in $D_3^{(m)}$.
  Then,

$$Q_{D_3^{(m)}} = \frac{l_1l_2l_3}{(l_1,l_2,l_3)}.\frac{l_4l_5l_6}{(l_4,l_5,l_6)} \cdots \frac{l_{3m-2}l_{3m-1}l_{3m}}{(l_{3m-2},l_{3m-1},l_{3m})}$$

  We can obtain the basis criteria for generalized spline modules on Friendship graph $D_3^{(m)}$ over any GCD domain as follows:

- **Theorem**
  Let ($D_3^{(m)}$,α) be an edge labeled Friendship graph with 2m+1 vertices $v_1, v_2, \ldots, v_{2m+1}$ and $3m$ edge labels $l_1 \ldots, l_{3m}$. It is obtained by joining m copies of triangles $T_1, T_2, \ldots, T_m$ together along the common vertex v_1, which is cut vertex in $D_3^{(m)}$. Let $\{F_1, ..., F_{2m+1}\} \subset R_{(D_3^{(m)},\alpha)}$,then $\{ F_1, \ldots, F_{2m+1} \}$ forms a basis for $R_{(D_3^{(m)},\alpha)}$ if and only if

$$\mid F_1F_2 \ldots F_{2m+1} \mid = Q_{D_3^{(m)}} = r.Q_{T_1}Q_{T_2} \ldots Q_{T_m}$$

  where r $\in$ R is a unit.

Extending the above result, basis criteria for generalized spline modules on Dutch windmill graph $D_n^{(m)}$ over any GCD domain can be proved as follows

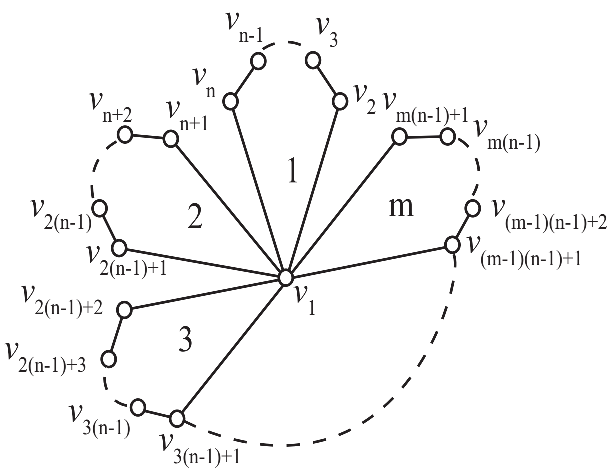


FIG. 7.16: Dutch windmill graph,$D_n^{(m)}$

- **Corollary**

Let $(D_n^{(m)}, \alpha)$ be an edge labeled Dutch windmill graph with $m(n-1)+1$ vertices $v_1, v_2, \ldots, v_{m(n-1)+1}$ and mn edge labels l_1,. . .,l_{mn}.

It is obtained by joining m copies of n-cycles C_{n_1},C_{n_2},. . ., C_{n_m} together along the common vertex v_1, which is cut vertex in $D_n^{(m)}$.

Then from the definition of Q_G [8] for any graph G, we have

$$Q_{D_n^{(m)}} = \frac{l_1 l_2 \ldots l_n}{(l_1, l_2, \ldots, l_n)} . \frac{l_{n+1} \ldots l_{2n}}{(l_{n+1}, \ldots, l_{2n})} \ldots \frac{l_{mn-(n-1)} \ldots l_{mn}}{(l_{mn-(n-1)}, \ldots, l_{mn})}$$

Then $\{F_1,...,F_{m(n-1)+1}\}$ forms a basis for $R_{(D_n^{(m)}, \alpha)}$ if and only if

$$\mid F_1 F_2 \ldots F_{m(n-1)+1} \mid = Q_{D_n^{(m)}} = r.Q_{C_{n_1}}.Q_{C_{n_2}} \ldots Q_{C_{n_m}}$$

where r $\in$ R is a unit.

Now we consider Complete graph K_4 and Wheel graph W_4 (Fig.7.17(a) and (b)) which are isomorphic to each other. These graphs have no common cut vertices with Cycle graphs, Diamond Graphs or Trees. We found basis criteria for generalized spline modules on these two isomorphic graphs separately over GCD domain.

From the definition of Q_G [8], for any graph G, we give Q_{K_4} as

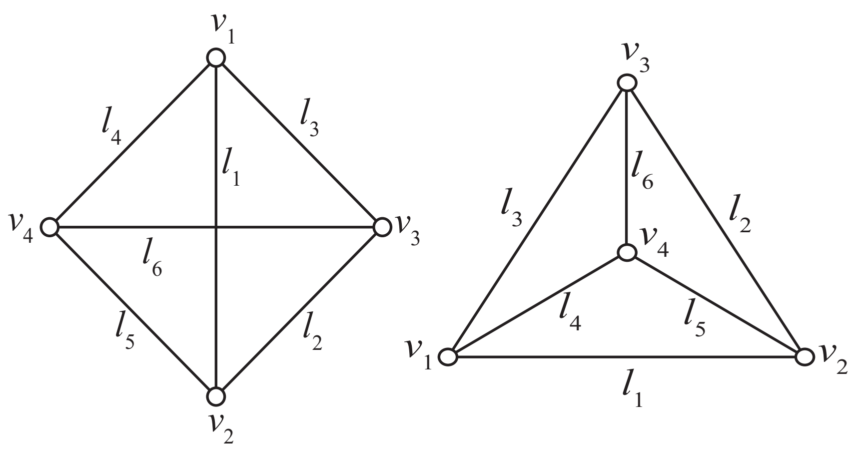


FIG. 7.17: (a) Complete Graph K_4 (b) Wheel graph W_4

$$\begin{aligned}
Q_{K_4} &= [l_1,(l_2,l_3),(l_4,l_5),(l_2,l_6,l_4),(l_5,l_6,l_3)]\cdot[l_2,l_3,(l_6,l_4),(l_6,l_5)]\cdot[l_4,l_6,l_5] \\
&= \frac{l_1(l_2,l_3)(l_4,l_5)(l_2,l_4,l_6)(l_3,l_5,l_6)}{(l_1,(l_2,l_3),(l_4,l_5),(l_2,l_4,l_6),(l_3,l_5,l_6))}\cdot\frac{l_2l_3(l_6,l_4)(l_5,l_6)}{(l_2,l_3(l_6,l_4),(l_6,l_5))}\cdot[l_4,l_5,l_6] \\
&= \frac{(l_4,l_5)(l_5,l_6)(l_6,l_4)[l_4,l_5,l_6]l_1l_2l_3(l_2,l_3)(l_2,l_4,l_6)(l_3,l_5,l_6)}{((l_1,l_2,l_3,(l_4,l_5)),(l_2,l_4,l_6,(l_3,l_5,l_6)))(l_2,l_3,l_6,l_4,(l_6,l_5))} \\
&= \frac{l_4l_5l_6(l_4,l_5,l_6)l_1l_2l_3(l_2,l_3)(l_2,l_6,l_4)(l_5,l_6,l_3)}{((l_1,l_2,l_3,l_4,l_5),(l_2,l_4,l_6,l_3,l_5))(l_2,l_3,l_6,l_4,l_6,l_5)} \\
&\qquad [\ (l_4l_5)(l_5,l_6)(l_6,l_4)[l_4,l_6,l_5] = l_4l_5l_6(l_4,l_5,l_6)] \\
&= \frac{l_1l_2l_3l_4l_5l_6(l_2,l_3)(l_2,l_4,l_6)(l_3,l_5,l_6)(l_4,l_5,l_6)}{((l_1,l_2,l_3,l_4,l_5),(l_2,l_3,l_4,l_5,l_6))(l_2,l_3,l_4,l_5,l_6)}
\end{aligned}$$

The basis criterion for edge labeled Complete graph is as follows:

- **Theorem**

  Let (K_4,α) be an edge labeled Complete graph. Let $\{ F_1, F_2, F_3, F_4 \} \subset R_{(K_4,\alpha)}$. Then

  $\{ F_1, F_2, F_3, F_4 \}$ is a basis for $R_{(K_4,\alpha)}$, if and only if $|\ F_1\ F_2\ F_3\ F_4\ | =$ r Q_{K_4}, where r $\in$ R is a unit.

  We found basis criterion for generalized spline modules on Wheel graph W_4 with 4 vertices, which is isomorphic to Complete graph K_4 by calculating Q_{W_4} using the above method.

  From the definition of Q_G[8], for any graph G, we have

$$\begin{aligned}
Q_{W_4} &= [l_1,(l_2,l_3),(l_4,l_5),(l_2,l_6,l_4),(l_5,l_6,l_3)].[l_2,l_3,(l_6,l_4),(l_6,l_5)].[l_4,l_6,l_5] \\
&= \frac{l_1l_2l_3l_4l_5l_6(l_4,l_5,l_6)(l_2,l_3)(l_2,l_6,l_4)(l_5,l_6,l_3)}{((l_1,l_2,l_3,l_4,l_5),(l_2,l_3,l_4,l_5,l_6))(l_2,l_3,l_4,l_5,l_6)}
\end{aligned}$$

which is equal to Q_{K_4}

Now, we can have the following theorem for basis criterion for generalized spline modules on Wheel graph with 4 vertices over any GCD domain.

- **Theorem**

Let (W_4,α) be an edge labeled Wheel graph . Let $\{F_1, F_2, F_3, F_4\} \subset R_{(W_4,\alpha)}$. Then $\{ F_1, F_2, F_3, F_4 \}$ is a basis for$R_{(W_4,\alpha)}$, if and only if $| F_1\ F_2\ F_3\ F_4 | =$ r Q_{W_4}, where r $\in$ R is a unit.

Here we observed that Complete graph K_4 and Wheel graph W_4 which are isomorphic to each other have same set of smallest leading entries for their flow-up splines. Also, the formula for Q_G is equal for these graphs and basis criterion for generalized spline modules on these graphs is same, over any GCD domain.This led us to study of generalized splines on isomorphic graphs as discussed in the following subsection.

The following is the fourth major result of our research study.

- **Basis Criteria for Generalized Spline Modules on Some Isomorphic Graphs**

We observed that graphs which are isomorphic to each other have same or equivalent basis criteria since zero trails of these graphs are same and thus Q_G is also same for these graphs. We proved that the basis criterion for generalized spline modules on each graph of an arbitrary set of isomorphic graphs is same over any principal ideal domain .This result is based upon the result proved in [7] which says that flow-up basis exists for generalized spline modules on arbitrary graphs over any PID.

Also, we have studied basis criteria for generalized splines on some isomorphic graphs over GCD domain and constructed flow-up basis for generalized spline modules on an arbitrary tree. An algorithm is developed for indexing the vertices of an ordered rooted tree graph such that the above method can generate a flow-up basis for tree graph and its isomorphic graphs.

First,we have given the basis criterion for generalised spline modules on a set of isomorphic graphs, over any GCD domain R.

- **Theorem:**

Let $\{G_1, G_2, \ldots, G_k\}$ be a set of isomorphic graphs. Then the basis criterion for generalized spline modules on each of these graphs, if exists, is same over any GCD domain.

We give an example to show that the cycle graph C_5 and it's isomorphic graph $C_5^{'}$ have the same $Q_G = \dfrac{l_1l_2l_3l_4l_5}{(l_1, l_2, l_3, l_4, l_5)}$, as calculated using the zero trails.

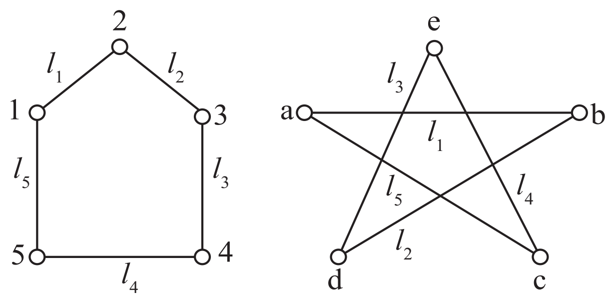


Fig. 7.18: (a)Cycle graph C_5 (b)Isomorphic graph C_5'

- **Example:**

  We know that the set $\{F_1, F_2, F_3, F_4, F_5\} \in R_{(C_5,\alpha)}$ forms a basis if and only if $|\,F_1F_2F_3F_4F_5| = rQ_{C_5}$,
  where $r \in R$ is a unit.
  Since, C_5' is isomorphic to C_5, Q_{C_5} will be the same as $Q_{C_5'}$, and hence the images $\{F_1', F_2', F_3', F_4', F_5'\}$ will form a basis for $R_{(C_5',\alpha')}$.

  Now we discuss the method of indexing of vertices of an ordered rooted tree so that flow-up basis can be constructed for generalized spline modules over these trees.It follows from Theorem on basis criteria of isomorphic graphs that all isomorphic trees will have the same basis criteria over any GCD domain. First we discuss the method of indexing of vertices of a star tree with 6 vertices.

  We define the flow-up classes of generalized splines for the star graph with six vertices, using the zero trail method.

- **Star Tree with 6 vertices**:

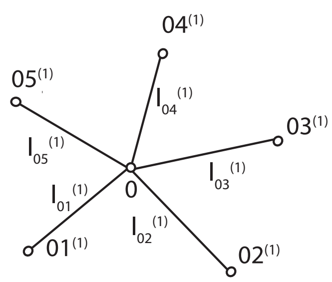


Fig. 7.19: Star Tree with 6 vertices

The indexing of the vertices is done level-wise. The root vertex is indexed as 0, and all the five leaf vertices at level 1 are indexed as $01^{(1)}$, $02^{(1)}$, $03^{(1)}$, $04^{(1)}$ and $05^{(1)}$ respectively in anticlockwise sense. Thus, any generalized spline over this graph can be expressed as

$$P = \begin{bmatrix} p_{05}^{(1)} \\ p_{04}^{(1)} \\ p_{03}^{(1)} \\ p_{02}^{(1)} \\ p_{01}^{(1)} \\ p_0 \end{bmatrix}$$

Here $p_v \in R$ is the vertex label corresponding to the v^{th} vertex in the graph.The flow-up classes for this graph $\{F^0, F^{01(1)}, F^{02(1)}, F^{03(1)}, F^{04(1)}, F^{05(1)}\}$ are obtained as follows:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ l_{01}^{(1)} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ l_{02}^{(1)} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l_{03}^{(1)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ l_{04}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} l_{05}^{(1)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Clearly, each class in the set $\{F^0, F^{01(1)}, F^{02(1)}, F^{03(1)}, F^{04(1)}, F^{05(1)}\}$ satisfies the GKM or edge condition and hence is a generalized spline over the star graph with six vertices. Also, we can see that the determinant $|\ F^0\ F^{01(1)}\ F^{02(1)}\ F^{03(1)}\ F^{04(1)}\ F^{05(1)}\ |$ $= l_{01}^{(1)}\ l_{02}^{(1)}\ l_{03}^{(1)}\ l_{04}^{(1)}\ l_{05}^{(1)} = Q_G$, where G is the star graph with six vertices in this case. Hence, we conclude from theorem [2.14] in [8] that the set $\{F^0, F^{01(1)}, F^{02(1)}, F^{03(1)}, F^{04(1)}, F^{05(1)}\}$ forms a basis for the generalized spline module $R_{(G,\alpha)}$ for this graph. Next, we generalize the above method to arbitrary rooted tree graphs in which the vertices are ordered from left to right at all levels. Consider the ordered rooted tree with seven vertices .

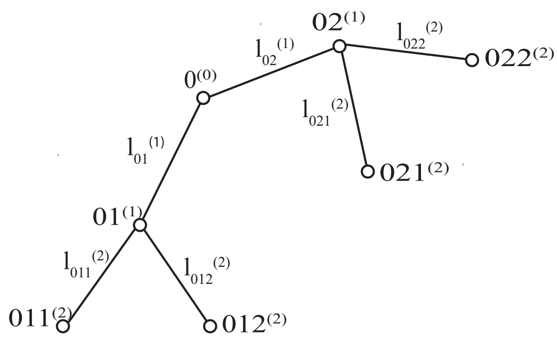


Fig. 7.20: Ordered rooted tree with 7 vertices

The root vertex is indexed as 0. There are two vertices in level 1, which are indexed as $01^{(1)}$ and $02^{(1)}$, ordered from left to right. The four leaf vertices are indexed as $011^{(2)}$, $012^{(2)}$ (children of vertex $01^{(1)}$ and $021^{(2)}$, $022^{(2)}$(children of vertex $02^{(1)}$,with the left to right ordering).

Any generalized spline over this graph can be expressed as

$$P = \begin{bmatrix} p_{022}^{(2)} \\ p_{021}^{(2)} \\ p_{012}^{(2)} \\ p_{011}^{(2)} \\ p_{02}^{(1)} \\ p_{01}^{(1)} \\ p_0 \end{bmatrix}$$

Using the zero trail method, we get the flow-up classes of generalized splines for this graph as $\{F^0, F^{01(1)}, F^{02(1)}, F^{011(2)}, F^{012(2)}, F^{021(2)}, F^{022(2)}\}$ which is equal to

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l_{01}^{(1)} \\ l_{01}^{(1)} \\ 0 \\ l_{01}^{(1)} \\ 0 \end{pmatrix} \begin{pmatrix} l_{02}^{(1)} \\ l_{02}^{(1)} \\ 0 \\ 0 \\ l_{02}^{(1)} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ l_{011}^{(2)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ l_{012}^{(2)} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ l_{021}^{(2)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} l_{022}^{(2)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We see that the determinant of the matrix
$\mid F^0 F^{01(1)} F^{02(1)} F^{011(2)} F^{012(2)} F^{021(2)} F^{022(2)} \mid = l_{01}^{(1)} l_{02}^{(1)} l_{011}^{(2)} l_{012}^{(2)} l_{021}^{(2)} l_{022}^{(2)} = Q_G$, for the graph G.

Thus, the set of generalized splines $\{F^0, F^{01(1)}, F^{02(1)}, F^{011(2)}, F^{012(2)}, F^{021(2)}, F^{022(2)}\}$ forms a basis for the module $R_{(G,\alpha)}$.

We have obtained the following algorithm for writing down the flow up basis for an arbitrary tree which is rooted and it's vertices at each level are ordered from left to right, which is as follows

- All entries for the flow up basis element F^0 are one.
- Let there be n vertices in level 1, indexed as $01^{(1)}$, $02^{(1)}$,. . ., $0n^{(1)}$. The ordering of these vertices are taken from left to right. Then the corresponding elements of flow up basis are $F^{01^{(1)}}$,$F^{02^{(1)}}$,. . ., $F^{0n^{(1)}}$, where $F^{0i^{(1)}}$ for $1 \leq i \leq n$ is

constructed by taking $F^{0i^{(1)}}_{0i} = F^{0i^{(1)}}_{0i1} = F^{0i^{(1)}}_{0i2} = \ldots = F^{0i^{(1)}}_{0in_i} = l^{(1)}_{0i}$, and all other entries as zero. Here $F^{0i^{(1)}}_{v}$ denotes the vertex label of the vertex v in the generalized spline $F^{0i^{(1)}}$ and $0i1, 0i2, \ldots, 0in_i$, are the children of the vertex 0i. This construction ensures that the GKM or edge conditions are satisfied by all the vertex labels of the spline $F^{0i^{(1)}}$.

– Similarly, the basis elements of the flow up basis corresponding to the children of the higher level vertices are constructed till we reach the leaf vertices. The leaf vertices will have only one non zero entry equal to the edge label of their parent vertices and zero otherwise.

It can be easily seen that the determinant of the matrix whose columns are the splines F^0,$F^{01^{(1)}}$,$F^{02^{(1)}}$,… is equal to the product of the edge labels of all edges in the tree graph and hence equal to Q_G, for the tree graph G. Thus the set of generalized splines $\{F^0, F^{01^{(1)}}, F^{02^{(1)}}, \ldots\}$ forms a flow up basis for G.

7.7 Conclusions

Our work is concluded by developing an algorithm to construct the generalized spline rings for the special graphs such as the complete graphs, complete bipartite graphs and hypercubes.

We found an algorithm for writing the generating set which acts as a basis for the generalized spline modules for cycle graphs and for wheel graphs, taking the base ring as quotient ring of integers.

We gave basis criteria for $R_{(G,\alpha)}$ on edge labeled Dutch windmill graph and special cases of Dutch windmill graph such as Friendship graph and Butterfly graph which have common cut vertices with Cycle graph C_n and triangles respectively, over any GCD domain by using determinantal techniques[8] and flow-up bases.

We have given basis criteria for complete graph K_4 and for wheel graph W_4, which are isomorphic to each other over any GCD domain. These graphs have no common cut vertices with cycle graphs, diamond graphs and trees.We observed that graphs which are isomorphic to each other have same or equivalent basis criteria since zero trails of these graphs are same and Q_G is also same for these graphs. We generalize this result and prove that basis criteria for generalized spline modules on each graph of an arbitrary set of isomorphic graphs is same over any principal ideal domain.Depending upon the type of graph and the base ring, R we can easily use this result to find Q_G as well as basis criteria for generalized spline modules on graphs which are isomorphic to some graphs like cycle graphs,diamond graphs, trees and Dutch windmill graphs.

We extended this result to generalized spline modules on isomorphic trees over any GCD domain and constructed Flow up basis for generalized spline modules on a star graph. An algorithm is developed for indexing the vertices of ordered rooted trees which helps us to generalize the method of constructing flow-up basis for generalized spline modules on any ordered rooted tree and hence on a family of isomorphic trees over a GCD domain.

7.8 Future directions for further research and open questions

The graphs we have used in our research find important applications in network and approximation theory and the present work adds to the existing knowledge and understanding in these and related areas. Also, it opens a vast field for research as we can think of studying the generalized spline moodules over these and other graphs by changing the base rings to other rings such as the polynomial rings and ring of Laurent polynomials. As these rings are PIDs, we can also try to find suitable bases for the generalized spline modules for these graphs.

We have basis criteria for generalized spline modules on arbitrary graphs over principal ideal domains.Further investigations on arbitrary graphs open a possibility of finding proof for general basis criteria for generalized spline modules on arbitrary graphs over any GCD domain.

Open Questions

- 1. Identify and study generalized spline modules on complete graphs,complete bipartite graphs,hypercube and cycle graph over polynomial rings and ring of Laurent polynomials.
- 2. Find proof of general basis criteria for generalized spline modules on arbitrary graphs over any GCD domain and give an algorithm to determine the entries of a flow-up class with the smallest leading entry on graphs like wheel graph,complete graph etc.