Chapter 1

Introduction

The term ‘splines’ meaning a ‘strip of wood or metal’ came from engineering, used by ship builders to construct models for the purpose of ship building. Lead weights called ’ducks’ were placed at key points by draftsmen so that the splines would bend, creating smooth curves in between the specified points. This usage of splines in shipbuilding was the earliest use of constructive geometry to draw free-form shapes and these techniques were used from 13th century to 16th century, before the invention of CAGD. Ship builders used to store these models in draftsman’s drawing board in the form of blueprints. Later classical drafting methods were combined with computational techniques and the classical drafting constructions were translated into numerical algorithms. Schoenberg’s fundamental paper in 1946 [96], introduced the theory of splines mathematically and became the focus of active research since then. Owing to their beautiful properties and wide applications in approximation theory and CAGD,spline theory became an important area for mathematicians,engineers and designers to delve deeper for better understanding of the subject. Beginning with the definition of splines as piecewise polynomial functions and restricting to polynomials in one dimension, we have

- **Polynomial spline[37]**

  Let the interval $[a, b]$ in $\mathbb{R}$ be subdivided into intervals $[t_i, t_{i+1}]$, where $i = 0, 1, 2, \ldots,$ $k - 1$, so that $[a, b] = [t_0, t_1] \cup [t_1, t_2] \cup \ldots \cup [t_{k-2}, t_{k-1}] \cup [t_{k-1}, t_k]$,
  $a = t_0 \leq t_1 \leq \ldots \leq t_{k-1} \leq t_k = b$

  The spline S is a function $S : [a, b] \longrightarrow \mathbb{R}$, where the restriction of S on each subinterval $[t_i, t_{i+1}]$ is a polynomial $P_i : [t_i, t_{i+1}] \longrightarrow \mathbb{R}[x]$,

  On the i^{th} sub interval of $[a, b]$,S is defined by P_i,

  $S(t) = P_0(t)$,$t_0 \leq t < t_1$,

$S(t) = P_1(t), t_1 \leq t < t_2,$

$\vdots$

$S(t) = P_{k-1}(t), t_{k-1} \leq t < t_k.$

The points $t_0, t_1, \ldots, t_k$ are called the knots or the vector $t=(t_0, t_1, \ldots, t_k)$ is called the knot vector, and if the polynomial pieces have degree at most n, then the spline S is of degree $\leq n$ or of order $n+1$. If $S \in C^{r_i}$ in a neighbourhood of t_i, then the spline is said to be of smoothness (at least) C^{r_i} at t_i, i.e, the pieces P_{i-1} and P_i have equal derivatives upto order r_i at t_i. Given a knot vector t, a degree n and a smoothness vector $r = (r_1, r_2, \ldots, r_n)$ for t, the set of splines of degree $\leq n$, equipped with the operations of pointwise addition and multiplication of functions and also taking real multiples of functions becomes a vector space. This space is commonly denoted as $S_n^r(t)$.

Initially piecewise polynomials were studied for curve-fitting, in fact, the word spline was only used for a particular C^2 interpolatory piecewise cubic polynomials. Later, the definition was broadened to include any piecewise polynomial. Besides the applications of splines in curve-fitting, they are widely used in the finite element method to estimate solutions of ordinary and partial differential equations. Subsequently, applied mathematicians and engineers working in the areas of curve fitting, finite element methods, computer-aided geometric design, signal processing, mathematical modelling, computer-aided design, computer-aided manufacturing, and circuits and systems started using multivariate splines extensively. In fact, spline functions are most successful approximating functions for practical purposes till today. In regression models, regression splines have several benefits when compared to linear and polynomial regressions. Unlike polynomial interpolation, which must use a high degree polynomial to produce flexible fits, splines introduce flexibility by increasing the number of knots, keeping the degree fixed. By their definition, the study of spline functions involves both algebra and geometry, and the smoothness conditions also require an understanding of analysis.

The cubic polynomial spline function represents the mathematical equivalent of the draftsman's wooden beam. S.Coons got recognition for his work[29] in the transition of aircraft drawings to computations. Later the theory of splines as an area of mathematical studies was extended in many different directions and first successful extension was made by C. De Boor [36],[37]. Many efforts have been made over several decades in developing the mathematics of spline functions, because of the importance of splines in industrial design. Smoothing splines have been used in fitting curves to data with the availability of algorithms to work on smoothing splines started in the late 1960s[90].

- **Example**[90]

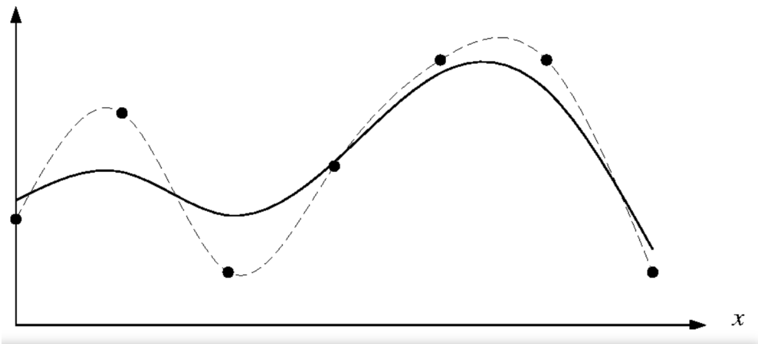


FIG. 1.1: A cubic interpolating spline—the dotted path—and a cubic smoothing spline—the continuous path.

Polynomial splines were extended to B-splines (Basis splines), which are sums of lower-level polynomial splines.These B-splines were introduced by I. Schoenberg for the case of uniform knots[96] in 1970s.The following figure shows an example of a cubic interpolating spline.

Basis splines are defined as

- **Basis Splines**[37]

B-splines are commonly used as basis functions to fit smoothing curves to large data sets. Given a non-decreasing knot vector,$(t_0, t_1, \ldots, t_{n+k-1})$,the n-basis splines of order k is defined by,

$$B_{i,1}(x) = \begin{cases} 1, & t_i \le x < t_{i+1} \\ 0, & \text{else} \end{cases}$$

$$B_{i,k}(x) = \frac{(x - t_i)}{(t_{i+k-1} - t_i)} B_{i,k-1}(x) + \frac{(t_{i+k} - x)}{t_{i+k} - t_{i+1}} B_{i+1,k-1}(x)$$

for $i = 0, \ldots, n-1$.The common case of cubic B-splines is given by $k = 4$.The above recurrence relation can be evaluated in a numerically stable way by the De Boor algorithm[37].

If we define appropriate knots on an interval $[a, b]$, then the B-spline basis functions form a complete set on that interval.Therefore we can expand a smoothing function as

$$f(x) = \sum_{i=0}^{n-1} c_i B_{i,k}(x)$$

given enough $(x_j, f(x_j))$ data pairs.The coefficients c_i can be readily obtained from a least squares fit.

1.1 Classical theory of splines

With the extension to higher dimensions, the C^r splines were defined over the polytopal complex $\Delta \subset R^d$ as $F = (F_\sigma)$ of polynomials, where $F_\sigma \in R = R[x_1, x_2, \ldots, x_n]$, for every n-dimensional face $\sigma_n \in \Delta_n$.Also if $\sigma \cap \sigma^{'} = \tau \in \Delta_{n-1}$, then $l_\tau^{r+1} | F_\sigma - F_{\sigma'}$, where l_τ is the linear form vanishing on τ. The set of all C^r splines on Δ was denoted by $S^r(\Delta)$ and $S_d^r(\Delta)$ consisted of all $F \in S^{r(\Delta)}$ with $deg(F_\sigma) \leq d$ for all $\sigma \in \Delta_n$. It was proved in [16] that $S_d^r(\Delta)$ was a subring of the product ring R^t as well had the module structure over the ring R. Many mathematicians started investigating the ring theoretic properties of $S_d^r(\Delta)$ and several important results were established in this area. As a module over R, the major question was finding the dimension and basis of $S_d^r(\Delta)$, in terms of combinatorial and geometric information of $\Delta \in R^d$.Billera [18] ,found the dimension and basis for $S_d^0(\Delta)$, for all n when Δ was a simplicial complex. However, for Δ to be a polyhedral complex, it could not be evaluated even for $n = 2$ or 3. Alfeld and Schumaker [3]found $dimS_d^r(\Delta)$ for $d \geq 3r + 1$, when $r > 0$. Schenck [94],[95] conjectured that $dimS_d^r(\Delta)$ was given by Schumaker's lower bound [3] for $d \geq 2r + 1$. Another question in this area was freeness of $S^r(\Delta)$ as a R-module and finding generators for $S^r(\Delta)$ whenever it was free. Billera and Rose [15],[18] introduced the definition of splines over the dual graphs of polyhedral complexes. This approach was later studied by many mathematicians as McDonald[77] and Schenck[94],Rose [93] etc. Independently, splines were studied by geometers and topologists and expressed as equivariant cohomology of torics and other algebraic varieties as in the works of Brion [22], Schenck [94],[95]. Goresky, Kottwitz and MacPherson [58] developed a combinatorial construction of equivalent cohomology called the GKM theory, which can be used for any algebraic variety X with an appropriate torus action. It builds a graph G_X whose vertices are the T-fixed points of X and whose edges are the one dimensional orbits of X. The theory became a powerful tool in Schubert Calculus and representation theory as in Knutson and Tao[67], and also in other fields.

1.2 Algebraization of Splines

Recalling the definition of C^r-splines we have,

for a polyhedral complex P, a C^r spline on P is a piecewise polynomial function (a polynomial is assigned to each $d-$dimensional cell or face σ of P), such that two polynomials supported on $d-$faces which share a common $(d-1)-$face τ, meet with order of smoothness r along the common face. The set of splines of degree at most k and are of smoothness of order r is denoted by $C_k^r(P)$, is a vector space[17]. A C^r-spline is represented as a vector of polynomials $(f_1, f_2, \ldots, f_n)$, where each f_i is a polynomial of degree at most k. Multiplying the vector by a fixed polynomial f gives $(f.f_1, f.f_2, \ldots, f.f_n)$, which is again a C^r-spline. This means that the set of splines is a module over the polynomial ring as shown in [18]. For two $d-$cells σ_1 and σ_2 sharing a common $(d-1)-$face τ, let l_τ be a nonzero linear form vanishing on τ. Billera and Rose [18] have shown that a pair of polynomials f_i supported on $\sigma_i, i = 1, 2$ meet with smoothness of order r along τ iff $l_\tau^{r+1} | f_1 - f_2$.

As an example, we see a 2$-$dimensional polyhedral complex which is a planar simplicial complex P and is the star of a single interior vertex v_0, the origin.The adjacent triangles or 2-faces meet over common lines, i.e, the 1$-$dimensional faces.

- **Example**[18]:

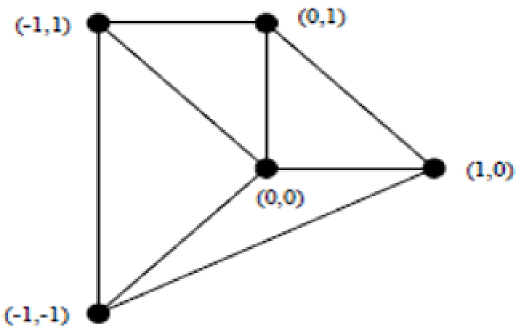


Fig. 1.2: Example of a C^r–spline

Begining with the simplex in the first quadrant and moving clockwise [Fig.1.2], the piecewise polynomials defined on the triangles are f_1, f_2, f_3, f_4.To obtain a global C^r function, we require element (f_1, f_2, f_3, f_4) to satisfy the conditions,

$a_1.y^{r+1} = f_1 - f_2;$
$a_2.(x - y)^{r+1} = f_2 - f_3;$
$a_3.(x + y)^{r+1} = f_3 - f_4;$
$a_4.x^{r+1} = f_4 - f_1.$

It can easily be verified that the continuity conditions are satisfied at every edge sharing the boundary of two simplexes[18].

Simcha Gilbert,Shira Polster and Juliana Tymoczko [55]expanded the family of objects on which the generalized splines were defined to arbitrary graphs which opened the possibilities of solving several open questions in classical theory of splines which could not be solved using other approaches. These splines were termed as generalized splines defined over edge labeled graphs in [55] as follows:

- **Edge Labeled Graph**[55]

  Let $G = (V, E)$ be a graph. Let R be an arbitrary commutative ring with identity which is also an integral domain and let I denote the set of all non-zero ideals of R. Let a function $\alpha : E \longrightarrow S$ be an edge labeling function defined on G,where α labels each edge in graph G by the ideals of the ring R. Then the graph G with function α is called an edge labeled graph which is denoted by $G = (V, \alpha)$.

  The definition of generalized splines over an arbitrary edge labeled graph G is as follows

- **Generalized Spline**[55]

  Let G = (V, E) be a graph of order n. Let R be a commutative ring and let I denote the set of all ideals of R. Let α : E $\longrightarrow$ I be an edge labeling. A generalized spline of (G, α) is a vertex labeling F : V $\longrightarrow$ R such that for each edge uv, $F(u) - F(v) \in \alpha(uv)$ where $F(u) \in R$ for each vertex in u in V.This condition is known as edge condition or GKM condition satisfied by the generalized splines over the edges of the graph G.The set of splines defined over G is denoted by $R_{(G,\alpha)}$. Each element of $R_{G,\alpha}$ is called a generalized spline. If the edge labeling is clear, it is denoted as R_G.

  The following figures (as discussed in [55]) are two examples of generalized splines R_{C_4} and R_{K_4}, defined on the 4-cycle C_4 and the complete graph K_4.

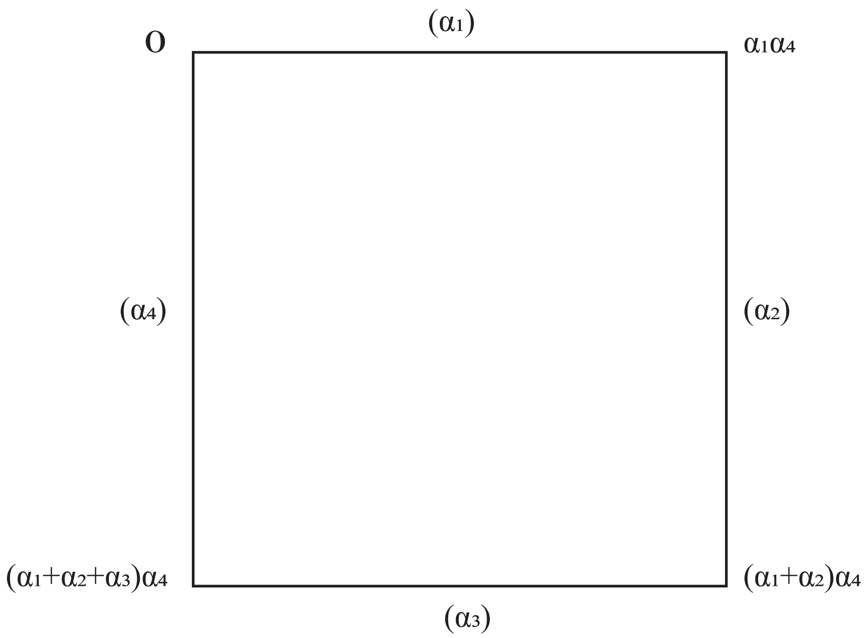


Fig. 1.3: Generalized spline on C_4

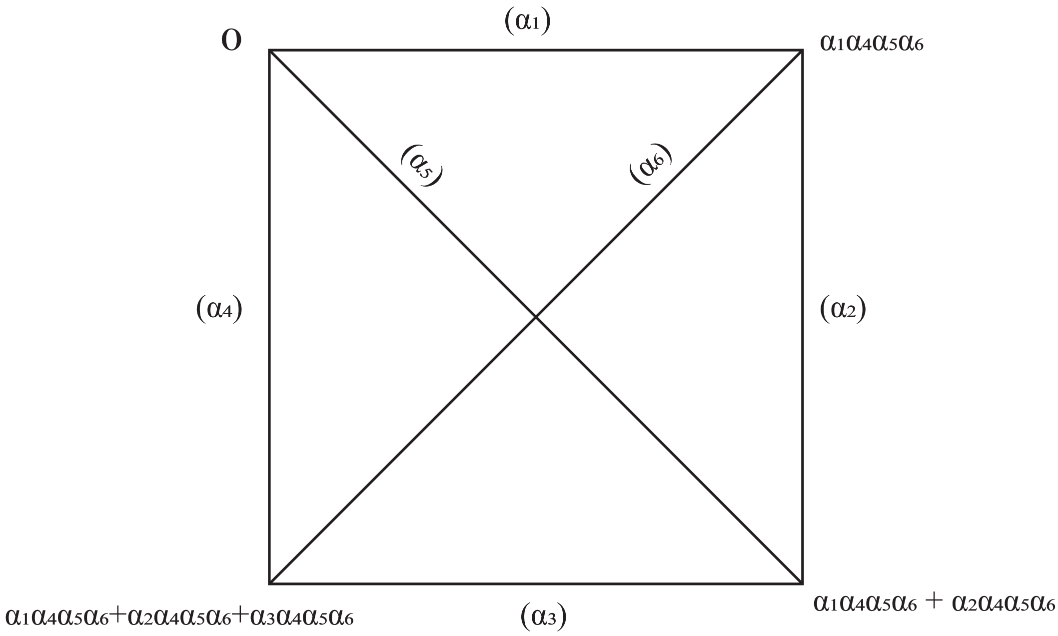


FIG. 1.4: Generalized spline on K_4

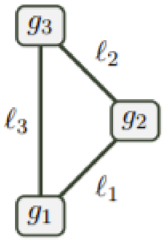
The set of generalized splines on an edge labeled graph has a ring structure and R-module structure like classical splines. Gilbert, Polster and Tymoczko [55] proved foundational results about the set of generalized splines, completely analysing the ring of generalized splines for trees. They have obtained the generalized splines for arbitrary cycles and have shown that the study of generalized splines for arbitrary graphs can be reduced to the case of different sub graphs, especially cycles or trees. They proved in [55] that every collection of generalized splines over an integral domain has free-submodule of rank $\mid V \mid$, producing a lower bound for the dimension of the ring of splines R_G, whenever R_G is a free module over R. Basic problems that arise naturally in the theory of generalized splines is that it focuses on particular cases of the choices of R,the graph G, the ring R and the edge labeling function α which maps the edges to the ideals of the ring R. Also, the module structure of the ring of generalized splines remains far from being understood in terms of freeness and existence of basis or generating set, for an arbitrary choice of the ring R[55]. Also,it is not clear how the ring R_G will be affected under the graph theoretic constructions such as addition or deletion of vertices.

A special type of generalized splines, which are called flow-up classes, are useful to find module bases for $R_{(G,\alpha)}$. Handschy, Melnick and Reinders [63] studied integer splines and the existence of flow-up classes on cycles over the ring of integers $\mathbb{Z}$. Each flow-up class has one more zero than its predecessor and the nonzero labels "flow up" the graph hence we write the flow-up class from bottom to top.If $G_k = (g_1, ..., g_n)$ is a flow-up class,then we write it as a vector[63].

$$G_k = \begin{bmatrix} g_n \\ \vdots \\ g_1 \end{bmatrix}$$

The following figures represent an example of a spline on a triangle,a system of congruences that it represents and also flow-up classes on a triangle.

- **Example**[63]



$$\begin{cases} g_1 \equiv g_2 \bmod \ell_1 \\ g_2 \equiv g_3 \bmod \ell_2 \\ g_3 \equiv g_1 \bmod \ell_3 \end{cases}$$

FIG. 1.5: Example of a spline on a triangle and the system of congruences

- **Example**[63]:

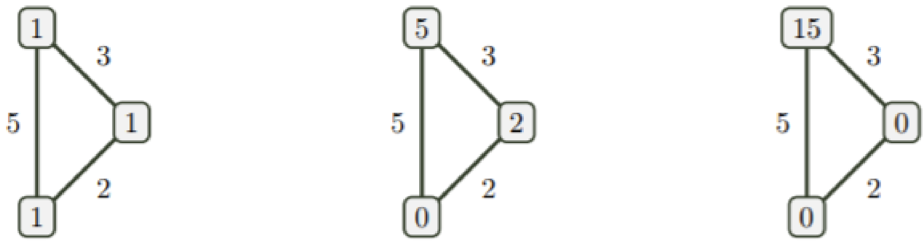


FIG. 1.6: Flow-up classes G_0, G_1 and G_2 on a triangle

Bowden and Tymoczko [21] considered the module of generalized splines over the quotient ring , $\mathbb{Z}/m\mathbb{Z}$ which is not an integral domain. They have shown that over $\mathbb{Z}/m\mathbb{Z}$, the minimum generating sets are smaller than expected. The ring $\mathbb{Z}/m\mathbb{Z}$ is a finite ring which is not an integral domain. Thus the generalized spline modules over $\mathbb{Z}/m\mathbb{Z}$ must have minimum generating sets namely a generating set with smallest possible size. The structure theorem for finite abelian groups [32]shows that finite modules are generally not free, but the minimum generating sets function like bases except that each element b of the minimum generating set has a scalar c_b satisfying $c_b.b = 0$.There are at most n elements in the minimum generating set of splines over the integers mod m which is not an integral domain, on a graph with n vertices [21]. The rank of the $\mathbb{Z}$-module of splines is defined to be the number of elements of a minimum generating set and Bowden ,Tymoczko proved that the smallest flow-up classes exist and formed a basis for generalized spline modules on a graph over $\mathbb{Z}/m\mathbb{Z}$. Nealy Bowden, Sarah Hagen and Stephanie Reinders [21] proved that flow-up classes with smallest leading entries form a module basis for $R_{(G,\alpha)}$, where R is an integral domain.Nealy Bowden, Sarah Hagen, Melanie King, and Stephanie Reinders[20] introduced two new bases for the module of integer generalized splines on cycles naming them as the triangulation basis and the King basis[20]. Each of these bases is fully expressible in terms of the edge labels of the cycles and the triangulation basis is constructed from triangulated cycles and hence exists for arbitrary cycles[20]. Their results generalized to principle ideal domains, which include classical univariate splines and Prufer domains[20].Also they presented the multiplication table of splines on cycles where the products of splines are expressed in terms of the King basis and found multiplication tables of equivariant cohomology rings in terms of Schubert bases which is the central problem of Schubert calculus[20].This work provides a criterion for the existence of flow-up bases[20].

In [50], Gjoni studied integer generalized splines on cycles and gave basis criteria for $\mathbb{Z}_{(C_n,\alpha)}$ via determinant of flow-up classes. Emmet Reza Mahdavi [74] characterized integer generalized splines on the diamond graph and developed a determinantal criterion for a given set of splines to form a basis.Also in [7], Selma Altinok and Samet Sarioglan proved the existence of flow-up bases for generalized spline modules on arbitrary graphs over principal ideal domains. They introduced a method [7] to determine the smallest leading entries of flow-up classes on arbitrary graph over a principal ideal domain, by using zero trails and gave an algorithm to determine flow-up classes on arbitrary ordered cycles. In [8], Selma Altinok and Samet Sarioglan generalized that work and gave basis criteria for R, where R is a GCD domain. They have given basis criteria for diamond graphs and trees over any GCD domain. They have also given basis criteria for graphs obtained by joining cycles, diamond graphs and trees together along common cut vertices.Katie Anders, Alissa S. Crans, Briana Foster-Greenwood,Blake Mellor and Juliana Tymoczo in [9] characterized the graphs that only admit constant splines for a large class of rings. They proved that if a graph has a particular type of cut set then the space of splines naturally decomposes as a certain direct sum of submodules.

In this study, we have obtained the generalized spline rings and the basis criterion for some very important family of graphs which find applications in network theory.

In Chapter 3, we have addressed some of the open questions posed by Simcha Gilbert, Shira Polster and Juliana Tymoczko in [55]. We have constructed the ring of generalized splines for the special cases, where G is a complete graph K_n, complete bipartite graph K_{n_1,n_2} and also for the hypercubes Q_n for all n, n_1, n_2. In all these graphs,the ring R is a commutative ring with identity which is also an integral domain and the edge labels are the non-zero ideals of the ring R. Also, the methods of constructing the generalized splines over the complete graphs K_n(for any n) and complete bipartite graphs K_{n_1,n_2} (for any n_1, n_2) have been generalized and Python code is developed to write these splines. The bipartite structure and Hamiltonicity of the hypercubes are used to find the general algorithm for writing the set of generalized splines R_{Q_n} (for any n).

In Chapter 4,we have obtained the basis criteria for $R_{(G,\alpha)}$ on edge labeled Dutch windmill graph and special cases of Dutch windmill graph such as Friendship graph and Butterfly graph, which have common cut vertices with Cycle graph C_n and triangles respectively, over any GCD domain.They have used zero trail method and formula Q_G given by Selma Altinok and Samet Sarioglan[7],[8].We have also seen that the results for the complete graph K_4 and the wheel graph W_4, which are isomorphic and have concluded that they have the same basis criteria over a GCD domain.

In Chapter 5,we extended the work done by Nealy Bowden and Julianna Tymoczko on cycles [21] to wheel graph which is a graph extension to cycle graph.We classified splines on wheel graphs, finding a minimum generating set of flow-up classes over $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime.We also classified splines on cycles over $\mathbb{Z}/m\mathbb{Z}$, if m has few prime factors and found a generating set of flow-up classes on these graphs over $\mathbb{Z}/m\mathbb{Z}$.

In Chapter 6,we proved that basis criterion for generalized spline modules on each graph of an arbitrary set of isomorphic graphs is same over any GCD domain, provided flow-up basis exists for those graphs.We have used results from [7],[8] and Chapter 5 for proving these results. We have constructed flow-up basis for generalized spline modules on an arbitrary tree over any GCD domain. Also an algorithm is developed for indexing the vertices of an ordered rooted tree graph using the results in [24] to establish isomorphism between trees..This helps us in obtaining the flow-up basis and basis criteria for an ordered rooted tree graph and all graphs that are isomorphic to these graphs.This work is submitted for publication.

Thus we have studied ring and module of generalized splines over a variety of graphs considering base rings which are either GCD domains,integral domains or quotient rings.However we have also realized that our study has generated several areas which can be taken up for further study and finding applications of the algorithms for the real world problems.