Chapter 2

Preliminaries

2.1 General Introduction

Billera and Rose [17],[18] pioneered the study of algebraic splines, introducing methods from commutative and homological algebra. Billera's solution of Strang's Conjecture (Theorem 5.8) in [15], using homological algebra indicated that the understanding and applicability of spline theory can be enhanced to a greater extent with this new approach. It became an active area of research and the study was enriched further by the research work carried forward by Ruth Haas, P. Alfeld, L. Schumaker, H. H. Andersen, John Morgan, Ridgway Scott, Hal Schenck etc.[62],[6],[5],[10],[78],[94].

Spline theory developed independently in topology and geometry. Simcha Gilbert, Shira Polster and Julianna Tymoczko [55], expanded the family of objects on which these splines were defined to arbitrary graphs, which they called the generalized splines. Their definition of generalized splines opened possibilities to do several things that weren't possible from the algebraic or geometric perspectives[55]. Their study was enriched by the works of Selma Altinok and Samet Sarioglan, Nealy Bowden, Sarah Hagen, Melanie King, and Stephanie Reinders, Polster. S and Tymoczko. J in [8],[7],[20],[21].

The objective of this research study is to delve deep into the study of generalized splines defined over a variety of graph families which are extensively used in network theory. In this chapter we give the preliminaries in commutative algebra, algebraic topology, graph theory and spline theory which we have used in our work. We have also included the important definitions and results discussed in the studies dealing with generalized splines, which are used in proving the results in our work. Although, the proofs of many theorems are excluded, but they can be obtained from the references given at the end.

In section 2.2 we have discussed the fundamentals in commutative algebra such as the definitions and examples of rings, subring, ideals, integral domains, quotient rings, product rings and ring homomorphism and isomorphism. In section 2.3 we have given the preliminaries from algebraic topology which includes polyhedral and simplicial complexes and their properties. In section 2.4,we have discussed algebraic approach to dimension problem of splines over polyhedral complexes which includes spline spaces over polyhedral complex and algebraic criterion followed by definitions of edge labeled graph, generalized splines,Isomorphism in edge labeled graphs, flow-up class,zero trails, Q_G for an edge labeled graph G, AHU Algorithm,isomorphism in ordered rooted trees and several examples for better understanding.Also we have given theorems,propositions,lemmas relevant to our work from [55],[63],[21],[7],[8].

2.2 Preliminaries from Commutative Algebra

- **Ring**

  A non-empty set R together with two binary operations $(+)$ and $(\cdot)$ called addition and multiplication respectively, is called a ring if it has the following three properties.

  (i) $(R, +)$ is an abelian group,i.e,

  (a)$\forall a, b \in R, a + b \in R$.

  (b) $\forall a, b, c \in R, (a + b) + c = a + (b + c)$.

  (c)$\exists 0 \in R$ such that $a + 0 = a = 0 + a$, $\forall a \in R$. 0 is unique and is called the additive identity or the zero element.

  (d) $\forall a \in R, \exists b \in R$ such that $a + b = 0 = b + a$. b is unique and is denoted by $-a$. It is called the additive inverse of a.

  (e) $\forall a, b \in R, a + b = b + a$

  (ii) $(R, \cdot)$ is a semi-group. i.e,

  (a) $\forall a, b \in R, a.b \in R$.

  (b) $\forall a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

  (iii) Distributive laws hold, i.e, $\forall a, b, c \in R, a.(b + c) = a.b + a.c$.

  The well known examples of rings are the following:

  Examples

  1. The sets of integers $(\mathbb{Z})$, rational numbers $(\mathbb{Q})$, real numbers $(\mathbb{R})$ and complex numbers $(\mathbb{C})$ with usual addition and multiplication are rings.

2. The set of all $n \times n$ matrices over $\mathbb{R}$ is a ring with respect to usual addition and multiplication of matrices.

Also, we have the polynomial rings which are defined as

- **Polynomial Ring**

Let R be a ring. The polynomial ring in X with coefficients in a ring R consists of formal expressions of the form: $g(X) = b_0 + b_1X + b_2X^2 + \ldots + b_mX^m$,where $b_i \in R$, $m \in N$. It can be easily verified that the ring theoretic properties are satisfied by the set of polynomials of degree atmost m, for $m \in N$.

We now give the definition of ideals in a ring.

- **Ideal**

Left ideal Let R be a ring. A subset I of R is called a left ideal of R if

1. I is a subgroup of (R, +), i.e., $a, b \in I \Rightarrow a - b \in I$ and

2. I is closed for arbitrary multiplication on the left by elements in R, i.e., $a \in I$ and $x \in R \Rightarrow xa \in I$.

Right ideal A subset I of R is called a right ideal of R if

1. $a, b \in I \Rightarrow a - b \in I$ and

2. $a \in I$ and $x \in R \Rightarrow ax \in I$.

Two sided ideal A subset I of R which is both a left ideal and a right ideal is called a two sided ideal, i.e.,

1. $a, b \in I \Rightarrow a - b \in I$ and

2. $a \in I$ and $x \in R \Rightarrow$ both $ax \in I$ and $xa \in I$.

Examples

1. R is an ideal in R and is called the unit ideal.(0) is also an ideal in R and is called the zero ideal.The ideals (0) and R are called the trivial ideals of R.

2.For a fixed integer n, $n\mathbb{Z} = \{nx | x \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$.

The principal ideal in a ring R is the ideal generated by a single element in R. The ideal $n\mathbb{Z}$ for any $n \in \mathbb{N}$ is a principal ideal in the ring of integers $\mathbb{Z}$. In fact, all multiples ax of an element $x \in R$ form a principal ideal denoted by (x)

It can be seen that every ideal in the ring of integers $\mathbb{Z}$ is principal ideal (m), generated by a single element $m \in \mathbb{Z}$.

- **Operations on Ideals**

If a, b are ideals in a ring R, their sum $a + b$ is the set of all $x + y$ where $x \in a$ and

$y \in b$. It is the smallest ideal containing a and b. More generally, we may define the sum $\Sigma_{i \in I} a_i$ of any family (possibly infinite) of ideals a_i, its elements are all sums $\Sigma_{i \in I} x_i$ where $x_i \in a_i$ for all $i \in I$ and almost all of the x_i (i.e. all but a finite set) are zero. It is the smallest ideal of R which contains all the ideals a_i. The intersection of any family (a_i) of ideals is an ideal. The product of two ideals a, b in R is the ideal ab generated by all products xy, where $x \in a$ and $y \in b$. It is the set of all finite sums $\Sigma x_i y_i$ where each $x_i \in a$ and $y_i \in b$. Similarly we define the product of any finite family of ideals. In particular, the powers $a^n (n > 0)$ of an ideal a are defined as, the ideal generated by all products $x_1 x_2 \ldots x_n$ in which each factor $x_i \in a$.

Examples

1)If $\mathbb{R} = \mathbb{Z}, a = (m), b = (n)$then $a + b$ is the ideal generated by the h.c.f of m and n and $a \cap b =$ null set $\iff m, n$ are coprime.

2)$R = k[x_1, \ldots, x_n]$, $a = (x_1, \ldots, x_n)$ be the ideal generated by $x_1, \ldots, x_n$.Then a^m is the set of all polynomials with no terms of degree $< m$.

Next we define a subring of a ring R.

- **Subring**
  Let R be a ring. A non-empty subset S of R is called a subring of R if (S , +) is a subgroup of (R, +) and (S, $\cdot$) is a sub semi-group of (R,$\cdot$). Or, equivalently, the restrictions of the operations (+) and (.) on R to S make (S, +, $\cdot$) a ring in its own right. It is obvious that a subring of a subring of a ring R is a subring of R.

  Examples

  1. The subsets 0 and R are subrings of any ring R and are called the trivial subrings of R.

  2. The subsets $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ are all subrings of $\mathbb{C}$.

  Next we give the definition and examples of a commutative ring with identity, division ring, quotient ring and integral domains.

- **Commutative ring with identity element**
  If the semi-group $(R,.)$ has an identity, it is unique, is denoted by 1 and is called the identity element of R. A ring R is said to be commutative if the semi-group $(R, .)$ is commutative,i.e,

  (a) $a \cdot b = b \cdot a, \forall a, b \in R$.

  (b)$\exists$ an identity element $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a, \forall a \in R$.

  Examples The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative with identity. Every non-zero element of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ is invertible and the inverse of $a \neq 0$ is $1/a$. However, the only invertible elements in $\mathbb{Z}$ are ± 1.

- **Division Ring**
  A ring R with multiplicative identity 1, where $1 \neq 0$ is called division ring if every nonzero element has multiplicative inverse,i.e, there exists $b \in R$ such that $ab = 1$.

  The rings $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are some examples of division rings.

  Units in a ring are defined as

- **Unit**
  Let R be a ring with identity $1 \neq 0$. An element u of R is called a unit in R if there is some v in R such that $uv = vu = 1$. Thus every non zero element in a division ring is a unit.

  Example
  Units of the ring $\mathbb{Z}$ of integers are ± 1.

  The zero divisors are the elements in a ring defined as

- **Zero divisor**
  An element $a \in R$ is said to be a left zero divisor if there exists $b \neq 0$ such that $a \cdot b = 0$. Similarly, a is a right zero-divisor if there is a $c \neq 0$ such that $c \cdot a = 0$. An element $a \in R$ is said to be a zero-divisor if a is either a left zero-divisor or a right zero-divisor.

  In any ring R with atleast two elements, 0 is a zero-divisor, called the trivial zero-divisor.

- **Integral Domain**
  A non-zero ring R is called an integral domain if there are no non-trivial zero-divisors in R. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are examples of integral domains.

  A subgroup allows us to partition a group into disjoint subsets of the same size, called cosets.Cosets are defined as,

- **Cosets**
  Let $H = \{h_1, h_2, h_3, \ldots\}$ be a subgroup of G.Then for $g \in G$ the set $gH := \{gh_1, gh_2, gh_3, \ldots\} = \{gh | h \in H\}$ is called the left coset of H with representative g. Analogously,the right coset with representative g is defined as $Hg := \{h_1 g, h_2 g, h_3 g \ldots\} = \{hg | h \in H\}$. The coset $eH = H = He$ is called the trivial coset of H.

- **Quotient Ring**
  Let R be a ring and I be a two sided ideal. Considering just the operation of addition, R is a group and I is a subgroup. Infact, since R is an abellian group under addition, I is a normal subgroup, and the quotient group R/I is defined as

$\{a + I : \forall a \in R\}$.The elements of R/I are known as cosets of I in R. Addition of cosets is defined by adding coset representatives:

$(a + I) + (b + I) = (a + b) + I$ for $a, b \in R$.

The zero coset is $0 + I = I$,and the additive inverse of a coset is given by $-(a + I) = (-a) + I$ However, R also comes with a multiplication,which is defined as R/I is a ring by multiplying coset representatives:

$(a + I).(b + I) = ab + I$ for $a, b \in R$

The following theorem [32] states that R/I is a ring, known as the quotient ring.

Theorem[**32**]
If I is a two sided ideal in a ring R , Then R/I has the structure of a ring under coset addition and multiplication.

An example of a quotient ring is the ring of integers modulo n which is defined as

- **Integers modulo** n
  For a fixed positive integer n, let $\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}$, be the set of remainders of integers modulo n. Under the addition and multiplication modulo n, $\mathbb{Z}_n$ is a ring, called the ring of integers modulo n.

  We note that the quotient ring $\mathbb{Z}_6$ is not an integral domain because the product of non zero elements 2 and 3 is equal to zero.

  As discussed earlier, a principal ideal is defined as,

- **Principal ideal**
  An ideal I in R is called a principal ideal if I is generated by one element, which is denoted by $I = (x)$ for some $x \in I$.

- **Principal ideal domain**
  A ring is called a principal ideal ring if R is commutative and every ideal of R is principal. A principal ideal ring which is an integral domain is called a principal ideal domain (PID).

  Thus, a commutative integral domain R is called a principal ideal domain (PID) if every ideal of R is principal, i.e, generated by one element.

  Example The ring of even integers $R = 2\mathbb{Z}$ is an example of a PID.

  We now define modules over a commutative ring R as follows

- **Module**
  Let R be a Commutative ring. An R-module is a pair (M,μ) where M is an

abelian group and μ is a mapping of $R \times M$ into M such that, if we write rm for $\mu(r,m)(r \in R, m \in M)$, the following axioms are satisfied:

$r(m_1 + m_2) = rm_1 + rm_2,$

$(r_1 + r_2)m = r_1m + r_2m,$

$(r_1r_2)m = r_1(r_2m)$ and

$1m = m(r_1, r_2, r \in R; m_1, m_2, m \in M)$

Example
An ideal I of R is an R-module. In particular, R itself is an R-module.

Sub modules and product modules are defined as

- **Sub module**
  A subgroup N of an R-module M is called an R sub module of M if $rn \in N$ for every $r \in R, n \in N$.

- **Direct product**
  Let R be any commutative ring and M, N be R-modules, then the Cartesian product $M \times N$ can be made into an R-module, called the direct product of M and N, in a natural way, $R \times (M \times N) \longrightarrow M \times N, (a,(x,y)) \longrightarrow (ax, ay)$.

  Special cases 1. $R^2 = R \times R$.

  2. $R^n = R \times R \times \ldots \times R$.

- **Generating set**
  If x is an element of M, the set of all multiples $rx(r \in R)$ is a submodule of M, denoted by Rx or (x). If $M = \sum_{i \in I} Rx_i$,then x_i are said to be a set of generators of M, i.e, every element of M can be expressed as a finite linear combination of x_i with coefficients in R.

- **Free module**
  An R-module M is called free if M has generators x_i such that $\sum a_ix_i = 0$ implies $a_i = 0$ for all i. The set of x_i is called a basis, i.e, a generating set consisting of linearly independent elements of M.

- **Rank of a module**
  The cardinality of any finite generating set or basis is called the rank of module,i.e, the minimum number of generators (if it exists) of M is called the rank of M.

  Example
  Every vector space is a free module.A free abelian group is a free module over the ring $\mathbb{Z}$ of integers.

Next, we give the definitions of ring homomorphism and isomorphism and module homomorphism and module isomorphism.

- **Ring homomorphism and Ring isomorphism**
  Let R and S be two rings. A homomorphism ϕ from R to S is a map of sets $\phi : R \longrightarrow S$ such that for all $x, y \in R$

  1. $\phi(x + y) = \phi(x) + \phi(y)$
  2. $\phi(xy) = \phi(x)\phi(y)$
  3. $\phi(1_R) = 1_S$

  A ring homomorphism which is a bijection (one-one and onto) is called a ring isomorphism.

  Examples
  1.The inclusion $\mathbb{Z} \longrightarrow \mathbb{Q}$ is a ring homomorphism.

  2.The map $A[X] \longrightarrow A$ sending a polynomial p to p(1) with the coefficients in the ring A (or, more generally,$p \longrightarrow p(a)$ for some $a \in A$) is a ring homomorphism.

- **Module Homomorphism and Module Isomorphism**
  In algebra, a module homomorphism is a function between modules that preserves the module structures. Explicitly, if M and N are left modules over a ring R, then a function $f : M \longrightarrow N$ is called an R-module homomorphism or an R-linear map if for any x, y in M and r in R,
  $f(x + y) = f(x) + f(y)$,
  $f(rx) = rf(x)$.
  A module homomorphism is called a module isomorphism if it is a bijection.

2.3 Preliminaries from Algebraic Geometry

We start this section with the definitions of polyhedral and simplicial complexes. Classically the splines were defined over these complexes. Later on they were defined on the duals of simplicial complexes and then on arbitrary graphs.

We begin with the following basic definition of k-simplex and simplicial complex.

- **k-Simplex**
  Let $\{a_0, a_1, \ldots, a_k\}$ be a set of geometrically independent points x in $\mathbb{R}^n$. The k-dimensional simplex or k-simplex σ^k spanned by $\{a_0, a_1, \ldots, a_k\}$ is the set of all

points in $\mathbb{R}^n$, for which there exists non-negative real numbers $\lambda_0, \lambda_1, \ldots, \lambda_k$, such that $x = \Sigma_{(i=0)}^{k} \lambda_i x_i$ and $\Sigma_{(i=0)}^{k} \lambda_i = 1$. The numbers $\lambda_0, \lambda_1, \ldots, \lambda_k$ are the barycentric coordinates of the point x. The points $a_0, a_1, \ldots, a_k$ are the vertices of the simplex σ^k. Here we observe that a 0-simplex is a point, a 1-simplex is a closed line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron.

A simplex σ^k is a face of a simplex σ^n, $k \leq n$, if each vertex of σ^k is a vertex of σ^n. All the faces of σ^n other than σ^n itself are called the proper faces of σ^n. The simplex σ^n with the vertices $\{a_0, a_1, \ldots, a_n\}$ is denoted by $\langle a_0, a_1, \ldots, a_n \rangle$.

Example
The faces of a 2-simplex $\langle a_0, a_1, a_2 \rangle$ are the 2-simplex itself, the 1-simplexes $\langle a_0, a_1 \rangle$, $\langle a_1, a_2 \rangle$ and $\langle a_0, a_2 \rangle$ and the 0-simplexes $\langle a_0 \rangle$, $\langle a_1 \rangle$ and $\langle a_2 \rangle$. Two simplexes σ^m and σ^n are properly joined provided they do not intersect or the intersection $\sigma^m \cap \sigma^n$ is a face of both σ^m and σ^n.

- **Simplicial Complex**
A geometric or simplicial complex is a finite family K of geometric simplexes which are proper and have the property that each face of a member of K is also a member of K. The dimension of K is the largest non-negative integer r such that K has a r-simplex. The union of the members of K with the Euclidean topology is denoted by $\mid K \mid$ and is called the geometric carrier of $\mid K \mid$ or the polyhedron associated with K.

Example
The collection $K = \{\langle a_0 \rangle, \langle a_1 \rangle, \langle a_2 \rangle, \langle a_0 a_1 \rangle, \langle a_1 a_2 \rangle, \langle a_0 a_2 \rangle, \langle a_0 a_1 a_2 \rangle\}$ is a simplicial complex of dimension 2.

Let X be a topological space. If there is a geometric complex K whose geometric carrier $\mid K \mid$ is homeomorphic to X, then X is said to be a triangulable space and the complex K is called a triangulation of X. The concept of connectedness is an equivalence relation in a complex K whose equivalence classes are called the combinatorial components of K. The complex K is connected if it has only one combinatorial component.

Next, we give the definition of simplicial isomorphism.

- **Simplicial Isomorphism**
Let X and Y be any two compact subsets of $\mathbb{R}^d$ and $\mathbb{R}^{d'}$ respectively. Suppose Δ and Δ' are triangulations of X and Y respectively. A map $F : (X, \Delta) \longrightarrow (Y, \Delta')$ is said to be a simplicial map if F maps vertices of Δ into vertices of Δ', such that $\sigma = \langle v_0, v_1, \ldots, v_n \rangle$ is a simplex of Δ implies $F(\sigma) = \langle F(v_0), F(v_1), \ldots, F(v_n) \rangle$ is a

simplex of Δ'. A simplicial map $F : (X, \Delta) \longrightarrow (Y, \Delta')$ is said to be a simplicial isomorphism if it is invertible.

The concept of affine subspace of a Euclidean space $\mathbb{R}^d$ as discussed below was used by Billera in his work in [15],[17],[18]. By an affine form in $\mathbb{R}^d$, we mean a polynomial of degree one, i.e, $l = a_0 + a_1x_1 + \ldots + a_dx_d$ where $a_i \in \mathbb{R}$ and $(x_0, x_1, \ldots, x_d) \in \mathbb{R}^d$. The points where l vanishes are given by $0 = a_0 + a_1x_1 + \ldots + a_dx_d$ and they constitute a hyperplane in $\mathbb{R}^d$. Next, we give the generalized concept of simplicial complexes. A finite region in $\mathbb{R}^d$ which is bounded by a finite number of hyperplanes of $\mathbb{R}^d$ is called a convex polytope in $\mathbb{R}^d$. The dimension of a convex polytope P is the dimension of the smallest affine space of $\mathbb{R}^d$ which contains P.

- **Polyhedral Complex**
  A finite collection P of convex polytopes in $\mathbb{R}^d$ is said to be a polyhedral complex if the following conditions are satisfied:

  (i) Any face of a member of P is again a member of P.

  (ii) The intersection of any two members of P is a face of both the members.

  Note that a simplicial complex is a special case of a polyhedral complex where all the convex polytopes are simplexes. The dimension of the biggest convex polytope occurring in the polyhedral complex P is called the dimension of P.

  In the next section we discuss the work done by L. Billera, L Rose[15],[17],[18], S. Deo[40] and W. Whitely, where they have used the algebraic approach to solve the dimension problem related to the vector space generated by the splines over polyhedral complexes.

2.4 Algebraic approach to dimension problem of splines over Polyhedral complexes

In order to understand the importance of the dimension problem, we give a brief survey of the work done by the mathematicians mentioned above, which opened a vast area of research with unsolved problems to be worked upon. First, we discuss about the vector space formed by the splines defined over a polyhedral complex.

- **Spline Spaces over Polyhedral complex**
  Let P be a finite d-dimensional polyhedral complex embedded in $\mathbb{R}^d$, i.e, P is a decomposition of a compact region in $\mathbb{R}^d$ into convex polytopes. For a simplicial

complex, we use the notation Δ. For non-negative integer r we define $S^r(P)$ to be the set of all piecewise polynomials on P, which are smooth of order r, i.e, all functions $F : P \longrightarrow R$, such that

(i) $F|_\sigma$ is a polynomial for each $\sigma \in P$.

(ii) F is continuously differentiable of order r.

Here R is the polynomial ring $\mathbb{R}[x_1, x_2, \ldots, x_d]$.

Such functions are called splines.The set of all such splines over P is denoted by $S^r(P)$, where as $S^r_k(P)$ denotes the subset of $S^r(P)$, consisting of functions involving polynomials of degree at most k.

In this definition, F will be differentiable of order r at a point $p \in P$ if, for all d-faces $\sigma \in P$ containing p, all partial derivatives of $F|_\sigma$ upto order r agree at p. We will see that the differentiability condition on F translates into a purely algebraic condition, which is extensively used in the study of $S^r(P)$.

We now discuss some properties of $S^r(P)$ and $S^r_k(P)$. Given an ordering $\sigma_1, \sigma_2, \ldots, \sigma_t$ of the d-dimensional faces of P, $F \in S^r(P)$ can be represented as a t-tuple of polynomials in $R = \mathbb{R}[x_1, x_2, \ldots, x_d]$, i.e, $F = (f_1, f_2, \ldots, f_t)$, where each f_i is $F|_{\sigma_i}$. Thus $S^r(P)$ can be regarded as a subring of the product ring $R^t = R \times R \times \ldots \times R$ (t-copies), with respect to pointwise operations of addition and multiplication. The set $S^r_k(P)$ forms a finite dimensional vector space over R.

The problem of finding the dimension and computing a basis for $S^r_k(P)$ was first formally introduced by G. Strang [99], who traced it's history to a paper of R. Courant[30]. L. Billera and L. Rose wrote a series of papers [15],[16],[17],[18],[93] in which they have covered some of the important aspects of the dimension problem with the methods used from commutative and homological algebra. This led to the algebraization of the set $S^r(P)$, which we explain below.First we discuss the zero set of a set of polynomials and the ideal of a subset of $\mathbb{R}^d$.

If $T \subset R$ is any set of polynomials, the zero set of T is defined as $Z(T) = \{p \in \mathbb{R}^d, f(p) = 0$ for all $f \in T\ \}$. If $X \subset \mathbb{R}^d$ is any set, then the ideal of X is defined as $I(X) = \{f \in R : f(p) = 0,$for all $p \in X\}$. Next, we give the algebraic criterion, which algebraizes the concept of smoothness of spline functions.

- **Algebraic Criterion**

  Let P be a d-complex and let $F : P \longrightarrow R^t$ be a piecewise polynomial function. Then $F \in S^r(P)$ if and only if, for every pair of d- faces σ_1, σ_2 in P, $F|_{(\sigma_1)} - F|_{(\sigma_2)}$ lies in $I(\sigma_1 \cap \sigma_2)^{(r+1)}$.

Now, as mentioned earlier, $S^r(P)$ is a subring of the product ring R^t with the operations of pointwise addition and multiplication, i.e.,

$(f_1, f_2, \ldots, f_t) + (g_1, g_2, \ldots, g_t) = (f_1 + g_1, \ldots, f_t + g_t)$

$(f_1, f_2, \ldots, f_t).(g_1, g_2, \ldots, g_t) = (f_1.g_1, \ldots, f_t.g_t)$

Here we observe that $(f_1.g_1, \ldots, f_t.g_t) \in S^r(P)$, for $f_i g_i - f_j g_j = f_i(g_i - g_j) - g_j(f_i - f_j)$ which lies in $I(\sigma_1 \cap \sigma_2)^{(r+1)}$ by algebraic criterion.

In addition to the above operations, if scalar multiplication is defined as $g.(f_1, f_2, \ldots, f_t) = (g.f_1, g.f_2, \ldots, g.f_t)$, then $S^r(P)$ becomes a submodule of R^t, the free R-module of rank t.

As discussed earlier, Billera [15] pioneered the work on algebraic splines with the methods from commutative and homological algebra, in order to prove a conjecture made by Strang [99], regarding the dimension of $C^1_k(P)$, for a planar simplicial complex P. Further, mathematicians such as Schumaker, Billera and Rose, McDonald and Schenck [94],[3], [17],[18] extended the study to piecewise polynomials over polyhedral complexes and in abstract algebraic settings studied the invariants of modules such as freeness, computing coefficients of Hilbert Polynomials, identifying syzygies of the span of edge ideals or analyzing algebraic varieties associated to the piecewise polynomials. Billera and Rose [17] defined the piecewise polynomials over the dual graphs of the polyhedral complexes which were found to be equivalent to the piecewise polynomials defined over the hereditary complexes. Goresky–Kottwitz–MacPherson [58] developed the GKM theory, which builds a graph G_X whose vertices are the T-fixed points of X and edges are the one dimensional orbits of X. Each edge in this graph is associated with a principal ideal $\langle \alpha_e \rangle$ in a polynomial ring. With this as the starting point, S. Gilbert, S. Polster and J. Tymoczo [55] defined the generalized splines over arbitrary graphs which allowed to extend the study forward and establish results which were not possible from algebraic perspective. Considering a graph whose edges were labeled with the ideals of a commutative ring R with identity, edge labeled graphs and generalized splines were defined as follows

- **Edge labeled graphs**[55]
  Let $G = (V, E)$ be a graph. Let R be an arbitrary commutative ring with identity which is also an integral domain and let I denote the set of all non-zero ideals of R. Let a function $\alpha : E \longrightarrow I$ be an edge labeling function defined on G,where α labels each edge in graph G by the ideals of the ring R. Then the graph G with function α is called an edge labeled graph which is denoted by (G, α).

The compatibility condition defined on the edges and the generalized splines were defined by imposing the compatibility or the GKM condition at all the edges of the graph.

- **Generalized Splines** [55]
  Let $G = (V, E)$ be a graph of order n. Let R be a commutative ring and let I denote the set of all ideals of R. Let $\alpha : E \longrightarrow I$ be an edge labeling. A generalized spline defined over (G, α) is a vertex labeling $F : V \longrightarrow R$ such that for each edge $uv, F(u) - F(v) \in \alpha(uv)$ where $F(u) \in R$ for each vertex u in V.This condition is known as edge condition or GKM condition[58],[55] satisfied by the generalized splines over the edges of the graph G. The set of generalized splines defined over G is denoted by $R_{(G,\alpha)}$. If the edge labeling is clear, it is denoted as R_G.

  It has been proved that $R_{(G,\alpha)}$ is a ring in Proposition 2.4 in [55].

- **Theorem**[55]
  R_G is a ring with identity $\mathbf{1}$ defined by $\mathbf{1_v} = \mathbf{1}$ for each vertex $v \in V$.

  Proof By definition R_G is a subset of the product ring $\bigoplus_{v\in V} R$, so we need only confirm that the identity is in R_G and that R_G is closed under addition and multiplication. The operations are component-wise addition and multiplication since R_G is in $\bigoplus_{v\in V} R$.The identity in $\bigoplus_{v\in V} R$ is the generalized spline 1 defined by 1_v = 1 for each vertex $v \in V$. This satisfies the GKM condition at each edge because for every edge e = uv we have $1_u - 1_v = 0$ and 0 is in each ideal $\alpha(e)$. The set R_G is closed under addition because if $p, q \in R_G$ then for each edge $e = uv$ we have $(p+q)_u - (p+q)_v = (p_u + q_u) - (p_v + q_v) = (p_u - p_v) + (q_u - q_v)$ which is in $\alpha(e)$ by the GKM condition. Similarly, the set R_G is closed under multiplication because if $p, q \in R_G$ then for each edge $e = uv$ we have $(pq)_u - (pq)_v = (p_u q_u) - (p_v q_v) = (p_u q_u - p_v q_u) + (p_v q_u - p_v q_v) = q_u(p_u - p_v) + p_v(q_u - q_v)$ which is in $\alpha(e)$ by the GKM condition.

  It is proved [55] that R_G becomes a module over the ring R with the operation of coordinate-wise addition and scalar multiplication where multiplication by r $\in$ R, gives the element $rp = (rp_{v_1}, rp_{v_2}, \ldots, rp_{v_n}) \in R_G$.

  Figures 1.3 and 1.4 in chapter1 (discussed in [55]) are two examples of the ring of generalized splines R_{C_4} and R_{K_4}, defined on the 4-cycle C_4 and the complete graph K_4. Here, R is any commutative ring with identity and (α_e) denotes the ideal generated by the single ring element of R. Thus, $p = (0, \alpha_1\alpha_4, (\alpha_1 + \alpha_2)\alpha_4, (\alpha_1 + \alpha_2 + \alpha_3)\alpha_4) = (p_{v_1}, p_{v_2}, p_{v_3}, p_{v_4})$ represents a generalized spline for C_4, because the difference $p_{v_2} - p_{v_1} = \alpha_1\alpha_4 \in (\alpha_1)$, and similarly for other adjacent vertices. Another example giving a generalized spline for the complete graph K_4 is given in Fig.1.4 in chapter 1. Once again, a generalized spline on K_4 will be written as $p = (0, \alpha_1\alpha_4\alpha_5\alpha_6, \alpha_1\alpha_4\alpha_5\alpha_6 + \alpha_2\alpha_4\alpha_5\alpha_6, \alpha_1\alpha_4\alpha_5\alpha_6 + \alpha_2\alpha_4\alpha_5\alpha_6 + \alpha_3\alpha_4\alpha_5\alpha_6) = (p_{v_1}, p_{v_2}, p_{v_3}, p_{v_4})$ which satisfies the edge conditions for all pairs of adjacent vertices.

The following theorem proved in [55] gives the construction of non trivial generalised splines over the cycle graph C_n for any n.

- **Theorem**[55]

Let C_n be a finite edge labeled cycle, given by vertices v_1,v_2,. . .,v_n in order. Define the vector $p \in R^{|V|}$ with

$$\begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_{n-1}} \\ p_{v_n} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \alpha_{1,n} \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} + \begin{bmatrix} \alpha_{1,2} \\ \alpha_{2,3} \\ \alpha_{3,4} \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{n-2,n-1} \\ \alpha_{n-1,n} \end{bmatrix}$$

with arbitrary choices of $p_{v_1} \in R$, $\alpha_{i,i+1} \in \alpha(e_{i,i+1})$, and $\alpha_{1,n} \in \alpha(e_{1,n})$. Then p is a generalized spline for C_n. The spline p is nontrivial exactly when $\alpha_{1,n}$ and at least one of the $\alpha_{i,i+1}$ are non-zero.

In fact, if it can be written in the form

$$\begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_{n-1}} \\ p_{v_n} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \alpha_{1,n}\alpha_{1,2} \begin{bmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \alpha_{1,n}\alpha_{2,3} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \dots + \alpha_{1,n}\alpha_{n-1,n} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

with coefficients $p_{v_1} \in R$ and $\alpha_{i,i+1} \in \alpha(e_{i,i+1}) = I_{i,i+1}$ for all $1 \leq i \leq n-1$.

The vectors $[1, 1, 1, \dots, 1]^T, [1, 1, 1, \dots, 0]^T, \dots, [1, 0, \dots, 0, 0, 0]^T$ are linearly independent in R^n but are not necessarily elements of R_{C_n}.

If R is an integral domain then for fixed choices of $\alpha_{i,j} \in \alpha(e_{i,j}) = I_{i,j}$, the vectors $[1, 1, 1, \dots, 1]^T, \alpha_{1,n}\alpha_{1,2}[1, 1, \dots, 1, 0]^T, \dots, \alpha_{1,n}\alpha_{n-1,n}[1, 0, \dots, 0, 0, 0]^T$ are both linearly independent and in R_{C_n} .

The GKM condition, as given in [55] for the complete graph K_4, Fig.1.3 in chapter1[55], whose edge labels are all principal ideals $\alpha(e)$ are given as

$p_{v_1} - p_{v_2} \in \alpha(e_{1,2}) = \langle \alpha_1 \rangle$
$p_{v_1} - p_{v_3} \in \alpha(e_{1,3}) = \langle \alpha_5 \rangle$
$p_{v_1} - p_{v_4} \in \alpha(e_{1,4}) = \langle \alpha_4 \rangle$
$p_{v_2} - p_{v_3} \in \alpha(e_{2,3}) = \langle \alpha_2 \rangle$
$p_{v_2} - p_{v_4} \in \alpha(e_{2,4}) = \langle \alpha_6 \rangle$
$p_{v_3} - p_{v_4} \in \alpha(e_{3,4}) = \langle \alpha_3 \rangle$
Where $p = (p_{v_1}, p_{v_2}, p_{v_3}, p_{v_4})$ is a generalized spline in K_4.

Further, Gilbert et. al.[55] have given the corollary (5.2) for the existence of flow-up basis for the generalized spline modules over an arbitrary edge labeled graph G, whenever the base ring R is a Principal ideal domain, which is as follows

- **Corollary**[55]
  Let R be an integral domain and (G, α) a connected edge–labeled graph on n vertices. Then R_G contains a free R-submodule of rank n.

  The next corollary (5.4) from [55]is very important with respect to our work. It is as follows

- **Corollary**[55]
  If G contains any subgraph G' for which $R_{G'}$ contains a nontrivial generalized spline,then R_G also contains a nontrivial generalized spline.

  The above corollary is used to construct the generalized splines for the edge labeled graph (K_4, α), using the generalized splines for the graph (C_4, α), as C_4 is a subgraph of K_4. The construction is as given in the following example:

- **Example**[55]

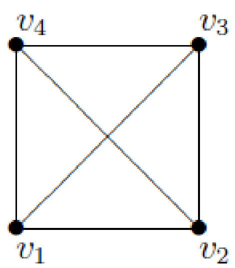

FIG. 2.1: GKM Conditions for K_4 whose ideals are all principal

We can construct generalized splines for the edge–labeled graph (K_4, α) given in Fig.2.1 [55] using these corollaries. Let C_4 denote the Hamiltonian cycle determined by ordering the vertices v_1, v_2, v_3, v_4. Let

$N_{C_4} = lcm\{\alpha(v_1v_3), \alpha(v_2v_4)\}$ with the labeling in Fig.2.1.

$$p = \begin{bmatrix} 0 \\ \alpha(v_1, v_4)\alpha(v_1, v_2) \\ \alpha(v_1, v_4)(\alpha(v_1, v_2) + \alpha(v_2, v_3)) \\ \alpha(v_1, v_4)(\alpha(v_1, v_2) + \alpha(v_2, v_3) + \alpha(v_3, v_4)) \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_4\alpha_1 \\ \alpha_4(\alpha_1 + \alpha_2) \\ \alpha_4(\alpha_1 + \alpha_2 + \alpha_3) \end{bmatrix}$$

The corollaries show that the multiple $N_{C_4}.p$ is a generalized spline for K_4.

We have constructed the generalized splines for the complete graph K_n for any $n \geq 4$, the complete bipartite graphs K_{n_1,n_2}, for $n_1, n_2 \geq 0$ and for hypercubes Q_n for any n.

Further, the isomorphism for edge labeled graphs is defined in [55]as

- **Isomorphism in Edge labeled Graphs** [55]
  Let (G, α) and (G', α') be edge labeled graphs and R be a commutative ring. A homomorphism of edge labeled graphs $\phi : (G, \alpha) \longrightarrow (G', \alpha')$ is a graph homomorphism $\phi_1 : G \longrightarrow G'$ together with a ring automorphism $\phi_2 : R \longrightarrow R$, so that for each edge $e \in E_G$, we have $\phi_2(\alpha(e)) = \alpha'(\phi_1(e))$.

$$\begin{tikzcd} E_G \arrow[r, "\phi_1"] \arrow[d, "\alpha"'] & E_{G'} \arrow[d, "\alpha'"] \\ I \arrow[r, "\phi_2"] & I \end{tikzcd}$$

An isomorphism of edge–labeled graphs is a homomorphism of edge–labeled graphs whose underlying graph homomorphism is an isomorphism.

The isomorphism of edge labeled graphs also establishes the isomorphism between the generalized spline rings as is proved in Proposition 2.7 in [55]

- **Proposition**[55]
  If $\phi : (G, \alpha) \longrightarrow (G', \alpha')$ is an isomorphism of edge–labeled graphs then ϕ induces an isomorphism of the corresponding rings of generalized splines $\phi_* : R_G \longrightarrow R_{G'}$ defined as $\phi_*(p)_{\phi_1(u)} = \phi_2(p_u)$ for each u $\in V_G$.

Further, a special type of splines called the flow-up classes was introduced by Handschy, Melnick and Reindeer in [63], which was used to find module bases for $R_{(G,\alpha)}$ .The flow-up class is defined as

- **Flow-up Class** [8]

  A flow-up class $F^{(i)}$ on an edge labeled graph with n vertices where $1 \leq i \leq n$, is a special class of generalized splines in $R_{(G,\alpha)}$, with the components $F_i^{(i)} \neq 0$ and $F_j^{(i)}$ $= 0$ for all $j < i$. The spline in the flow-up class contains $(i-1)$ leading zeroes.The set of all i-th flow-up classes is denoted by F^i.

  As discussed in [7], an example of flow up classes on the graph G, with the edge labeling as shown in Fig.2.2.

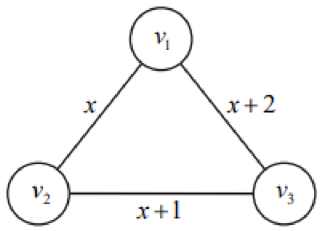

FIG. 2.2: Example of flow-up spline

Let (G, α) be as in the Fig.2.4. Example of flow-up classes on (G, α) can be given as $F^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $F^{(2)} = \begin{bmatrix} -x-2 \\ x \\ 0 \end{bmatrix}$, $F^{(3)} = \begin{bmatrix} x^2+3x+2 \\ 0 \\ 0 \end{bmatrix}$

N.Bowden,S.Hagen, M.King and S.Reinders in [20] has proved the following theorem

- **Theorem**[20]

  Let $\mathbb{R}$ be the ring of integers and (G, α) be an edge labeled graph with n vertices. The following statements are equivalent

  (a) The set $\{F^{(1)}, F^{(2)}, \ldots, F^{(n)}\}$ forms a flow-up basis for $R_{(G,\alpha)}$.

  (b) For each flow-up class $G^{(i)} = (0, \ldots, 0, g_i, g_{i+1}, \ldots, g_n)$, the entry g_i is a multiple of the entries in $F_i^{(i)}$.

  As it is easy to check the existence of the flow-up classes and the above theorem gives the importance of the leading entries of these classes in determining whether they form a basis.

  Selma Altinok and Samet Sarioglan [7], has introduced trails known as the zero trails in an edge labeled graph, which were used for determining the leading entries of flow-up classes over an integral domain R. The trails and zero trails were defined as

- **Zero Trails**[7]

  Let $G = (V, E)$ be a graph with an edge labeling α. Let $u, v \in V$. A $u - v$ trail in

G is an alternating sequence $T = (\; u = v_{i_0},\, e_{i_1},\, v_{i_1},\, \ldots \;,\, e_{i_k}, v_{i_k} = v)$ of vertices and edges such that $e_{ij} = v_{i(j-1)}v_{ij}$ and all the edges in T are distinct. If $\alpha(e_{ij}) = l_{ij}$, then the trail T is denoted by $l_{i_1},\, l_{i_2}, \ldots, l_{i_k}$. If $v_{i_k} = 0$, then T is called a zero trail and is denoted by $T^{(u,0)}$. Also, gcd and lcm of $\{\, l_{i_1},\, l_{i_2}, \ldots,\, l_{i_k}\}$ are denoted by $(\,T) \; = (\, l_{i_1}, l_{i_2}, \ldots, l_{i_k})$ and $[\,T\,] = [\, l_{i_1},\, l_{i_2}, \ldots,\, l_{i_k}\,]$ respectively.

They have given the following example of zero trails

- **Example**[7]:

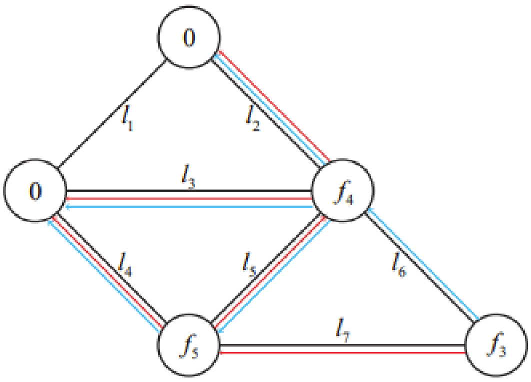

FIG. 2.3: Zero Trails

Let (G_1,α) be the edge labeled graph (Fig.2.3) and $(0, 0, f_3, f_4, f_5) \in F^{(3)}$, where $F^{(3)}$ is the flow-up class as discussed before. The zero trails of v_3, which is labeled as f_3 are shown as red and blue lines in Fig.2.3.

The zero trails of v_3 are listed below: $p_1^{(3,0)} = l_7l_4, p_2^{(3,0)} = l_7l_5l_3, p_3^{(3,0)} = l_7l_5l_2, p_4^{(3,0)} = l_6l_3, p_5^{(3,0)} = l_6l_2, p_6^{(3,0)} = l_6l_5l_4$

The set of all greatest common divisors of zero trails of v_3 is given as: $\{(p^{(3,0)})\} = \{(l_7, l_4), (l_7, l_5, l_3), (l_7, l_5, l_2), (l_6, l_3), (l_6, l_2), (l_6, l_5, l_4)\}$.

Consider the spline conditions induced by zero trails. For instance, for the zero trail l_7l_4, we have the following conditions.

$f_3 \equiv f_5 \bmod l_7$,
$f_5 \equiv 0 \bmod l_4$.

It implies that $f_5 = k_4l_4$ and $f_3 = f_5 + k_7l_7 = k_4l_4 + k_7l_7$ for some $k_4, k_7 \in \mathbb{R}$.

Hence (l_7, l_4) divides $k_4l_4 + k_7l_7 = f_3$. This holds also for other zero trails of v_3.Selma Altinok and Samet Sarioglan have proved the following important theorem showing the existence of flow-up classes over a graph G, whenever R is a PID.

- **Theorem**[7]

Let (G, α) has n vertices and R be a PID. Fix v_i with $i > 1$ and assume that all vertices v_j with $j < i$ are labeled by zero. Then a flow-up class $F^{(i)}$ exists with the first nonzero entry $f_i = [p^{(i,0)}]$

A corollary to this theorem (corollary 3.9) is

- **Corollary**[7]
  Let (G, α) be an edge labeled graph with n vertices. If the base ring R is a PID, then there exists a flow-up basis $\{F^{(1)}, \ldots, F^{(n)}\}$ where $F_i^{(i)} = [\{(p^{(i,0)})\}]$ for $1 < i \leq n$ and $F^{(1)} = (1, \ldots, 1)$.

Although, the zero trail method cannot be used for huge graphs, but a basis criteria for a set of spline modules to become a basis for R_G is given by them in [7].

First, we discuss the matrix form of a set of splines in $R_{(G,\alpha)}$.

Let (G, α) be an edge labeled graph with n-vertices. Let $A = \{F_1, \ldots, F_n\} \in R_{(G,\alpha)}$ with F_i =$(f_{i_1}, \ldots, f_{i_n})$. We can rewrite A in a matrix form, whose columns are the elements of A such as

$$A = \begin{bmatrix} f_{1n} & f_{2n} \ldots & f_{nn} \\ \vdots & & \\ f_{12} & f_{22} \ldots & f_{n2} \\ f_{11} & f_{21} \ldots & f_{n1} \end{bmatrix}$$

and determinant $\mid A \mid$ is denoted by $\mid F_1 F_2 \ldots F_n \mid$. Selma Altinok and Samet Sarioglan have given basis criteria for $R_{(G,\alpha)}$ by using this determinant.They have defined the element $Q_G \in R$ as follows

- **Q_G for an edge labeled graph G** [8]
  Let (G,α) be an edge labeled graph with k vertices. Fix a vertex v_i on (G, α) with $i \geq 2$. Label all vertices v_j with $j < i$ by zero. By using the notations in [8]. Q_G is defined as $Q_G = \prod_{i=2}^{k}[\ \{(\ p_t^{(i,0)})\ \}|t = 1, \ldots, m_i]$,where $p_t^{(i,0)}$ are zero trails of v_i and m_i is the number of the zero trails of v_i.

With the above definition of Q_G, Q_{C_n} for the cycle graph C_n is computed and also basis criteria for $R_{(C_n,\alpha)}$ are given by Selma Altinok and Samet Sarioglan[8]. They have given the basis criterion for the module $R_{(G,\alpha)}$, for G to be tree graph over a GCD domain R using the determinantal techniques and flow-up basis. These results given by them in [8] are as follows

- **Lemma**[8]
  Let (C_n, α) be an edge labeled n-cycle. Then

  $Q_{C_n} = \dfrac{l_1 l_2 \ldots l_n}{(l_1, l_2, \ldots, l_n)}$ where $l_1, l_2, \ldots, l_n$ are edge labels of the cycle graph C_n.

- **Theorem** [8])
  Let (C_n,α) be an edge labeled n-cycle and let $\{F_1, \ldots, F_n\} \subset R_{(C_n,\alpha)}$. Then $\{F_1, \ldots, F_n\}$ forms a basis for $R_{(C_n,\alpha)}$ if and only if $\mid F_1 F_2 \ldots F_n \mid$ = r$\cdot Q_{C_n}$, where r $\in$ R is a unit.

  Here, the set $\mid F_1 F_2 \ldots F_n \mid$ represents the determinant of the matrix notation of the set $\{F_1, F_2, \ldots, F_n\}$.

  The formularizing of Q_G for a tree graph G is

- **Lemma** [8]
  Let G be a tree with n vertices and k edges. Then $Q_G = l_1 \ldots l_k$ where $l_1, l_2, \ldots, l_k$ represent the edge labels.

  The following theorem in [8] gives the basis criterion for the module $R_{(G,\alpha)}$ for G to be a tree graph, over a GCD domain R

- **Theorem** [8]
  Let G be a tree with n vertices and k edges. Then $\{F_1, \ldots, F_n\} \subset R_{(G,\alpha)}$ forms a basis for $R_{(G,\alpha)}$ if and only if $\mid F_1 F_2 \ldots F_n \mid = r \cdot Q_G$ where $r \in R$ is a unit and R is a GCD domain.

  They have also given the basis criteria for the graphs obtained by joining cycles, diamonds and trees along common cut vertices as in the following theorem

- **Corollary** [8]
  Let $\{G_1, \ldots, G_k\}$ be a collection of cycles, diamond graphs and trees and let G be a graph obtained by joining $G_1, \ldots, G_k$ together along common vertices which are cut vertices in G. Then $\{F_1$,...,$F_n\} \subset R_{(G,\alpha)}$ forms a basis for $R_{(G,\alpha)}$ if and only if $\mid F_1 F_2 \ldots F_n \mid = r\cdot\ Q_{G_1} \ldots Q_{G_k}$, where $r \in$ R is a unit.

  As we know that flow-up bases exists for $R_{(G,\alpha)}$ over an arbitrary graph G, whenever the base ring is a PID, the following result in [8] follows

- **Theorem**[8]
  Let (G, α) be an edge labeled graph with n vertices and R be a PID. Then $\{F_1, \ldots, F_n\} \subset R_{(G,\alpha)}$ forms a module basis for $R_{(G\alpha)}$ if and only if $\mid F_1 F_2 \ldots F_n \mid = r.Q_G$, where R is a GCD domain and $r \in R$ is a unit.

  Selma Altinok and Samet Sarioglan have given the following conjecture in [8] for an arbitrary graph G over R to be a GCD domain.

- **Conjecture**[8]
  Let (G, α) be any edge labeled graph with n vertices. Then $\{F_1, \ldots F_n\} \subset R_{(G,\alpha)}$

forms a module basis for $R_{(G,\alpha)}$ if and only if $\mid F_1 F_2 \ldots F_n \mid = r.Q_G$ where $r \in R$ is a unit.

In our work, we have extended the results of Selma Altinok and Samet Sarioglan to obtain the basis criteria over the Dutch windmill graph and it's special cases such as Friendship graph and Butterfly graph, over a GCD domain, by using the zero trail method and the basis criteria for the spline modules.

We have seen that if two graphs are isomorphic then the zero trails of their vertices are identical.As a result we have concluded that isomorphic graphs have equivalent bases and basis criteria, whenever generalized modules on these graphs are free or generating sets exist.The result does not have strong implications for arbitrary graphs over GCD domains as no polynomial-time algorithm exists for checking the isomorphism between graphs in general.However,as AHU algorithm exists for tree graph isomorphisms, our results are generalized over ordered rooted tree graphs which form a very important class of graphs in Computer Network Theory.

We now discuss the AHU algorithm for the isomorphism in trees.

- **AHU Algorithm**[25]
  This algorithm determines tree isomorphism in time $O(\mid V \mid)$ by associating a tuple with each vertex of a tree that describes the complete history of its descendants.

  The AHU [25] algorithm is a serialization technique for representing the vertices of a tree as unique string and is able to capture a complete history of a tree's degree spectrum and structure, ensuring a deterministic method of checking tree isomorphisms.In this algorithm, leaf nodes are assigned with a parenthesis "()". Every time we move upwards, we combine, sort and wrap the parentheses.

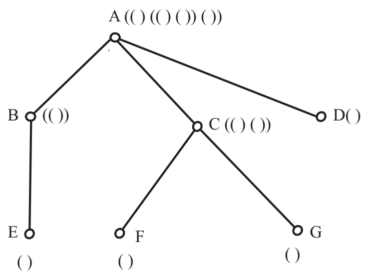

Fig. 2.4: Encoded ordered rooted tree using the AHU Algorithm

We can't process a node until we have processed all its children. We particularly consider the rooted trees because root of a tree represents the "start" of the data. Another important information assigned to the rooted trees is the ordering of the children (from left to right). Such a tree is called an ordered rooted tree.For example, suppose we have a tree with a single parent and two leaf nodes. So we assign "()" to

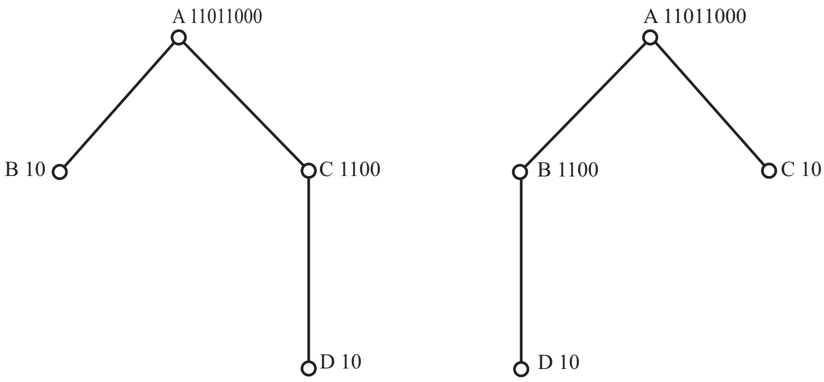

FIG. 2.5: Two isomorphic trees with same canonical names

the leaves. When we move towards the parent node, we combine the parentheses of leaves like "()()" and wrap it in another pair of parentheses like "(()())" and assign it to the parent. This process continues iteratively until we reach the root node [Fig.2.4].We can encode any ordered rooted tree by assigning a string of 0's and 1's, which uniquely determine the tree by using the AHU algorithm. In AHU algorithm, parenthetical tuples are assigned to all tree vertices. However, these parenthetical tuples have no ordering. Replacing "(" with "1" and ")" with "0", the parenthetical names are converted into canonical names, which can be sorted lexicographically.

Next we give the definition of isomorphism in ordered rooted trees.

- **Isomorphism in ordered rooted trees**[1]
  Two ordered rooted trees are isomorphic if there exists an isomorphism of rooted trees, such that it preserves the order of children of every vertex.The trees shown in Fig.2.5 are isomorphic.

  As discussed earlier, Gilbert et.al [55] has used the GKM matrix to describe all generalized splines for trees, for R to be a PID. The theorem 4.1 given by them is

- **Theorem** [55]
  Let $T = (V, E, \alpha)$ be a finite edge–labeled tree. The tuple $p \in R^{|T|}$ is a generalized spline $p \in R^{|T|}$ if and only if given any two vertices $v_i, v_j \in V$ we may write $p_{v_j} = p_{v_i} + \alpha_{i,i_1} + \ldots + \alpha_{i_{m-1},i_m} + \alpha_{i_m,j}$ for some $\alpha_{l,k} \in \alpha_{e_{l,k}} = I_{l,k}$ where $v_i, v_{i_1}, \ldots, v_{i_m}, v_j$ are the vertices in the unique path connecting v_i and v_j in the tree T. Furthermore p is non-trivial if and only if at least one of the $\alpha_{l,k}$ is nonzero.

  We have given an algorithm for indexing the vertices of an ordered rooted tree graph, which helps us in constructing the flow-up basis for such graphs and it's isomorphic graphs using the basis criteria for the spline modules for tree graphs over GCD domains.

We know from Gilbert's work [55] that the module of splines over a domain contains a free submodule of rank at least the number of vertices in the graph G and over a PID the module is always free with rank equal to the number of vertices. N Bowden and J. Tymokzko [21] have shown that a very different phenomena emerges if the base ring is considered as the quotient ring $\mathbb{Z}/m\mathbb{Z}$, which is not a domain. The spline modules over $\mathbb{Z}/m\mathbb{Z}$ are finite and hence have minimum generating sets, i.e, generating sets having the smallest number of elements. It follows from the structure theorem of finite abelian groups that finite modules may not be free but these minimum generating sets function as the bases except that each element b in the minimum generating set has a scalar c_b, such that $c_b.b = 0$. The rank of $\mathbb{Z}$-modules is defined as the number of elements in the minimum generating set. Bowden and Tymokzko [21] have shown that the rank of modules over $\mathbb{Z}/m\mathbb{Z}$ is smaller than that expected. They have given the following theorem.

- **Theorem**[21]
  Suppose that G is a graph with n vertices. Both over the integers and over the ring $\mathbb{Z}/m\mathbb{Z}$ the maximum rank of a ring of splines R_G is n.

  The next result[21] given by them constructs the modules with any rank k, $0 \leq k \leq n$, where n is the number of vertices in the graph G.

- **Theorem**[21]
  If m has at least two distinct prime factors then for each $n \geq 2$ and each i with $2 \leq i \leq n$ there exists an edge-labeled graph G on n vertices with rk $R_G = i$.

  The following result and it's corollary is proved in [21], characterizes the module of splines over the ring $\mathbb{Z}/(p^k\mathbb{Z})$, for a prime p.

- **Theorem**[21]
  Fix a zero divisor a in $\mathbb{Z}/m\mathbb{Z}$. Suppose all of the edges of C_n are labeled with powers of a so the set of edge labels is $\{a^{k_1}, a^{k_2}, a^{k_3}, \ldots, a^{k_n}\}$. Without loss of generality assume that a^{k_1} is the minimal power in the set and that a^{k_1} is the label on edge l_n. Then the following set generates all splines on C_n.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_1 \\ \vdots \\ l_1 \\ l_1 \\ 0 \end{pmatrix} \begin{pmatrix} l_2 \\ \vdots \\ l_2 \\ l_2 \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} l_i \\ l_i \\ \vdots \\ l_i \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} l_{n-2} \\ l_{n-2} \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} l_{n-1} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

- **Corollary**[21]
  Let C_n be the cycle on n vertices, let p be a prime number and let k be any positive integer. Then the splines on C_n over $\mathbb{Z}/p^k\mathbb{Z}$ are generated by the minimum generating set B in the above Theorem.

  The next theorem in the same paper characterizes the spline modules over the quotient ring $\mathbb{Z}/m\mathbb{Z}$, where m is as given in the theorem

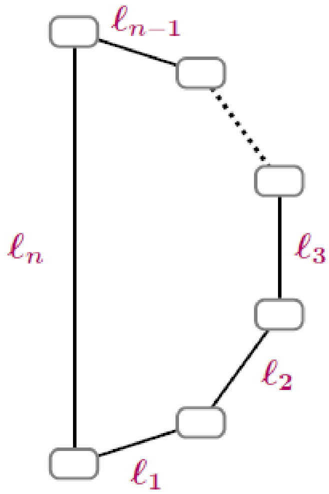

FIG. 2.6: Labeling conventions for general n-cycles

- **Theorem**[21]
  Let C_n be labeled as in Fig.2.6. Fix m, m_1, m_2 such that $m_1 \neq m_2$ and lcm $(m_1, m_2) = m$. Assume every edge of C_n is labeled with either m_1 or m_2 and that both m_1 and m_2 appear as edge labels at least once. Then the following set B is a flow-up generating set for R_{C_n}

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} z_1 \\ l_1 \\ \vdots \\ l_1 \\ l_1 \\ 0 \end{pmatrix} \begin{pmatrix} z_2 \\ \vdots \\ l_2 \\ l_2 \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} z_i \\ l_i \\ \vdots \\ l_i \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} z_{n-2} \\ l_{n-2} \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

where $z_i = 0$ if $l_i = m_2$ and $z_i = l_i$ if $l_i = m_1$