Chapter 3

An Algorithm for Generating Generalized Spline Modules on Graphs such as Complete graphs, Complete Bipartite Graphs and Hypercubes

3.1 Introduction

Gilbert, Juliana and Tymoczko [55] have expanded the family of objects on which the splines were defined to arbitrary graphs, labeling the edges of the graph by ideals of a commutative ring R. The splines were defined as the vertex labelings of the graph by the elements of R, such that the difference between the vertex labels of two adjacent vertices belonged to the corresponding edge label and have called these as generalized splines. Further they showed that the set of generalized splines formed a ring with the pointwise operations of addition and multiplication, inherited as a subring of the product ring R^t, for a commutative ring R. They have also proved the foundational result that nontrivial generalized splines can be defined over arbitrary graphs and have analyzed the ring of generalized rings for trees and cycle graphs. Several important results were proved in the subsequent research work carried out by Nealy Bowden and Julianna Tymoczko[21],Handschy, Julie Melnick and Stephanie Reinders[63], Bowden, Sarah Hagen, Melanie King, and Stephanie Reinders[20],etc. However, most of the research was focussed on the choice of particular family of graphs on which these splines were defined or specific rings such as integer rings or quotient rings. Questions regarding the identification of generalized spline rings on other families of graphs were left open. In this chapter, we have extended their study further, and have constructed the ring of generalized splines for the special graphs G, where G is a complete graph $K_n, n \geq 4$, complete bipartite graph K_{n_1,n_2}, where $(n_1, n_2) \in N$ and also the hypercube Q_n,where $n \geq 1$. The ring R is the commutative ring with identity, which is also an integral domain. We have also obtained the algorithms for generating the generalized splines for the above mentioned graphs and developed python codes for writing down these splines. The bipartite structure [52] and the hamiltonicity [49] of hypercubes are used to find the general algorithm to write down the generalized splines in R_{Q_n}, for any n.

In the next section, we give the important definitions and results that we have used in the results we have obtained.

3.2 Preliminaries

In this sub section, we give the fundamental results which describe the algebraic structure of the ring R_G, along with examples, which are used to construct new generalized splines for the complete graphs, complete bipartite graphs and hypercubes.The set of generalized splines on an edge labeled graph has a ring structure and R-module structure like classical splines. Gilbert, Polster and Tymoczko [55] proved some crucial results about the set of generalized splines, completely analysing the ring of generalized splines for trees. They discussed about the generalized spline ring R_G, for an arbitrary graph G over a commutative ring R, with the edge labels as the non-zero ideals of the ring R, in [55].They have obtained the generalized splines for arbitrary cycles and have shown that the study of generalized splines for arbitrary graphs can be reduced to the case of different sub graphs, especially cycles or trees.

Refering the definition of edge labeled graphs in chapter 2, the ring of generalized splines are defined as

- **Ring of Generalized splines**
  Let (G, α) be an edge–labeled graph. The ring of generalized splines is $R_{(G,\alpha)} = \{p \in \bigoplus_{v \in V} R$ such that p satisfies the GKM condition at each edge $e \in E\}$. Each element of $R_{(G,\alpha)}$ is called a generalized spline. When there is no risk of confusion, we write R_G.

  The definition of non-trivial generalized spline is as follows:

- **Definition**[55]:

  A nontrivial generalized spline is an element $p \in R_G$, that is not in the principal ideal **R1**, where **1** is the identity element in R_G defined as $\mathbf{1} = (1, 1, \ldots, 1)$.

  Basic problems that arise naturally in the theory of generalized splines is that it focuses on particular examples ,e.g, a particular choice of the ring R, the graph G and the edge labeling function α which maps the edges to the ideals of the ring R. Also, the module structure of the ring of generalized splines remains far from being understood in terms of freeness and existence of basis or generating set, for an arbitrary choice of the ring R[55]. Another aspect of the generalized spline ring which is not clear is how the ring R_G will be affected under the graph theoretic constructions such as addition or deletion of vertices. Exploring into these open areas, we could extend the study further and addressed the open question posed by Simcha Gilbert, Shira Polster and Juliana Tymoczko in [55]. We have constructed the ring of generalized splines for the special cases, where G is a complete graph K_n, complete bipartite graph K_{n_1,n_2} and also for the hypercubes Q_n. In all these graphs,the ring R is a commutative ring with identity which is also an integral domain and the edge labels are the non-zero ideals of the ring R. Also, the methods of constructing the generalized splines over the complete graphs K_n(for any n) and complete bipartite graphs K_{n_1,n_2} (for any n_1, n_2) have been generalized and Python code is developed to write these splines. The bipartite structure[52] and Hamiltonicity[49] of the hypercubes are used to find the general algorithm for writing the set of generalized splines R_{Q_n} (for any n). We first discuss the example of generalized spline ring R_{C_3} over the ring of integers for the cycle graph C_3.

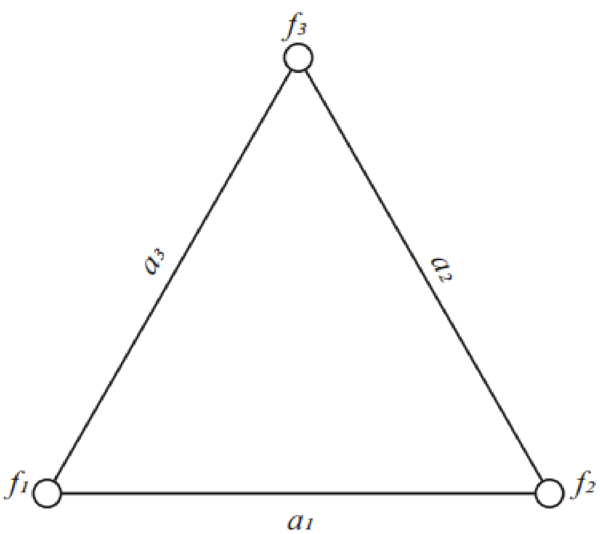

FIG. 3.1: Example of generalized integer spline on cycle graph C_3[47]

- **Example of generalized integer spline on cycle graph C_3[47]**

  Let the generalized integer spline $f = (f_1, f_2, f_3) \in R_{C_3}$, where C_3 is a 3-cycle with the edge labels a_1,a_2,a_3 where a_1,a_2 and a_3 are integers.

  The vertex labels (f_1, f_2, f_3) belonging to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ satisfy the following conditions
  $f_1 \equiv f_2 \bmod a_1$,
  $f_2 \equiv f_3 \bmod a_2$ and
  $f_3 \equiv f_1 \bmod a_3$.

Some of the important results for the generalized spline ring R_G, relevant to our work are mentioned in this subsection.We will refer to the preliminaries in the following subsection, throughout this chapter.The first result which is a corollary 5.4 to theorem 5.1 in [55],we get the condition for a generalized spline ring R_G to contain a non trivial generalized spline in terms of the generalized spline ring $R_{G'}$ where G' is a subgroup of G.It is as follows

- **Corollary**[55]

  If G contains any subgraph G' for which $R_{G'}$ contains a non-trivial generalized spline,then R_G also contains a nontrivial generalized spline.

- **Corollary**[55]

  Let R be an integral domain. If the graph G contains at least two vertices, then R_G contains a nontrivial generalized spline.

  With these and some results of [55], discussed in chapter 2, we obtain the generalized splines for the cycle graph C_3 which is also the complete graph K_3, to identify the generalized splines for the complete graph K_n, for $n \geq 4$.

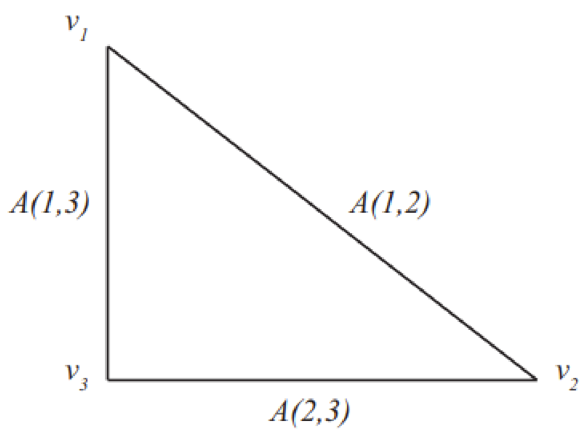

Fig. 3.2: Generalized spline on K_3

- **Generalized splines for Complete graphs,** $K_n, n \geq 3$

  Let us first consider the non-trivial generalized splines for complete graph K_3 [Fig.3.2]. Here the edges (v_1, v_2), (v_2, v_3) and (v_3, v_1) of the graph K_3 are labeled with the non-zero ideals $A(1,2), A(2,3)$ and $A(3,1)$ respectively of the ring R, when R is an integral domain.

  It follows from Theorem 3.8 in [55], a generalized spline p_{K_3} on the complete graph K_3 is

$$p_{K_3} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3) \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3) \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \end{bmatrix}$$

  Here we see that p_{K_3} satisfies the edge conditions on K_3, because if the vertices v_i and v_j are adjacent, then $p_{v_i} - p_{v_j} \in A(i,j)$,as $\alpha(i,j)$ is a factor of $p_{v_i} - p_{v_j}$. Here α(i, j) represents generator of the edge ideal $A(i,j)$. Let R_{K_3} denote the set of all generalised splines of (K_3, α). Since R is an integral domain and each $\alpha(i,j)$ is not equal to zero, R_{K_3} contains nontrivial generalized splines. Using the above result, we have generated the algorithm for developing the generalized spline for the complete graph K_n, for any $n \geq 4$.We first construct K_n from K_{n-1} by adding a new vertex v_n to K_{n-1} and then join this new vertex to the existing vertices $v_1, v_2, \ldots, v_{n-1}$ of K_{n-1}.

  Also, we will be using the edge conditions to identify the ring R_G, where graph G is complete bipartite graph K_{n_1,n_2}, for any n_1 and n_2.

  We have generated the algorithms for developing the generalized splines for complete graph K_n, for $n \geq 4$ and for complete bipartite graphs with the vertex sets V_1 containing n_1 vertices and V_2 containing n_2 vertices. We have used similar notations as above, where we denote the edge ideal corresponding to the edge joining the i^{th} and j^{th} vertices by $A(i,j)$ and $\alpha(i,j)$ represents an element of the non-zero ideal $A(i,j)$.We have also obtained the python codes for writing the generalized splines for the complete graphs and complete bipartite graphs.

  We have extended the method of writing algorithm for the generalized splines to hypercubes, Q_n, for $n \geq 2$ using the bipartite nature and hamiltonicity of the hypercubes, which find extensive use in coding theory. Hypercubes, denoted by Q_n, are graphs which find extensive use in coding theory in Computer Science and other areas of Mathematics.

3.3 Results and Discussions

Results and Discussions[47]

In this section, we first extend the method of constructing the ring of generalized splines R_{K_n}, for any $n \geq 4$, starting with the ring R_{K_3} for the complete graph K_3. In order to get the graph K_n, we add a new vertex to the graph K_{n-1} and join the new vertex to the existing $n-1$ vertices in K_{n-1}. In the following constructions we consider the ring R to be a commutative ring with identity and also an integral domain. First we construct the graph K_4 from the graph K_3 and obtain the ring of generalized splines R_{K_4} from the ring R_{K_3}.

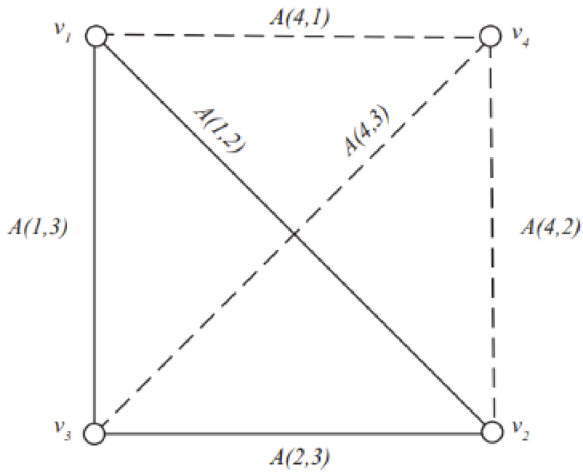

FIG. 3.3: Generalized spline on K_4[47]

We add the vertex v_4 to K_3 (Fig.3.3) and join the new vertex v_4 with the vertices v_1,v_2,v_3 of K_3. The new edges are labeled with the non-zero ideals $A(4,1), A(4,2), A(4,3)$ of integral domain R and $\alpha(4,1)$, $\alpha(4, 2)$, $\alpha(4, 3)$ are the generators of the of the respective edge ideals. It can be seen that every vertex label for $p_{K_3} \in R_{K_3}$ is multiplied by the factor $\alpha(4, 1)\alpha(4, 2)\alpha(4, 3)$ to get the corresponding vertex labels for the spline $p_{K_4} \in R_{K_4}$, where R_{K_4} denotes the set of all generalised splines for the edge labeled graph (K_4, α). It is easily verified that if the new vertex v_4 is labeled with p_{v_4} $=\alpha(4, 1)\alpha(4, 2)\alpha(4, 3)$, then p_{K_4} becomes a generalized spline for R_{K_4}, since the edge conditions are satisfied for the adjacent vertices in K_4. So we have

$$p_{K_4} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \\ \langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \end{bmatrix}$$

We can see that $p_{v_1} - p_{v_2} \in A(1,2)$, since $\alpha(1,2) \in A(1,2)$ is a factor of $p_{v_1} - p_{v_2}$

Similarly we have

$p_{v_2} - p_{v_3} \in A(2,3)$, since $\alpha(2, 3)$ is a factor of $p_{v_2} - p_{v_3}$
$p_{v_4} - p_{v_1} \in A(4,1)$, since $\alpha(4, 1)$ is a factor of $p_{v_4} - p_{v_1}$
$p_{v_4} - p_{v_2} \in A(4,2)$, since $\alpha(4, 2)$ is a factor of $p_{v_4} - p_{v_2}$
$p_{v_4} - p_{v_3} \in A(4,3)$, since $\alpha(4, 3)$ is a factor of $p_{v_4} - p_{v_3}$
Here $p_{v_4} = \alpha(4,1)\alpha(4,2)\alpha(4,3)$ is non-zero because R is an integral domain. Also, since K_3 is a sub-graph of K_4 and R_{K_3} contains nontrivial generalized splines, R_{K_4} also contains nontrivial generalized splines.This follows from the corollary 5.4 in [55],as already discussed in the begining of the chapter.

Using similar methods, we can identify the ring of generalized splines for the complete graph K_5.

- **Complete graph (K_5), n = 5[47]**

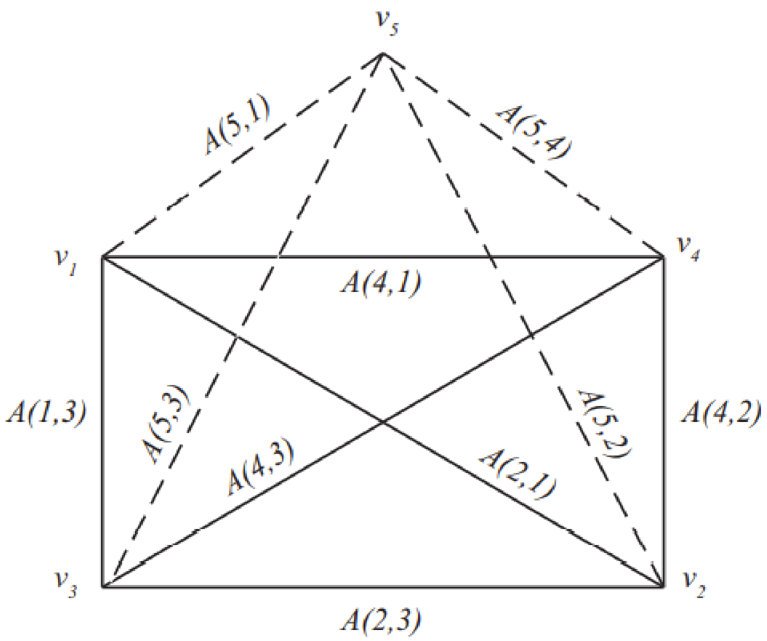

FIG. 3.4: Generalized spline on K_5

We can get K_5 by adding the vertex v_5 to K_4 and the four edges joining v_5 to the four vertices v_1 ,v_2 , v_3, v_4 of K_4 (Fig.3.4). Then in order to get any element of R_{K_5} , we multiply each element of R_{K_4} by α(5,1) α(5,2)α(5,3)α(5,4) and label the added vertex v_5 with the element α(5,1)α(5,2)α(5,3)α(5,4) $\in$ R. Then any element of R_{K_5} will be of the form p_{k_5} as given below.

$$p_{K_5} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle\langle\alpha(5,1)\alpha(5,2)\alpha(5,3)\alpha(5,4)\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle\alpha(4,1)\ldots\alpha(4,3)\rangle\langle\alpha(5,1)\alpha(5,2)\alpha(5,3)\alpha(5,4)\rangle \\ \langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle\langle\alpha(5,1)\alpha(5,2)\alpha(5,3)\alpha(5,4)\rangle \\ \langle\alpha(5,1)\alpha(5,2)\alpha(5,3)\alpha(5,4)\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \\ p_{v_5} \end{bmatrix}$$

Now, we give the algorithm for writing the generalized spline for complete graph K_n, for any n.

- **Theorem**[47]

We obtain the complete graph K_n by adding the n^{th} vertex v_n and the edges (v_n , v_1),(v_n , v_2),. . .,(v_n ,v_{n-1}) to the complete graph K_{n-1} . Labeling the new edges with the ideals $A(n,1), A(n,2)$,. . . $A(n,n-1)$, we get the generalized spline ring R_{K_n}, with the elements of the type

$$p_{K_n} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle N_4\rangle\langle N_5\rangle\ldots\langle N_n\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle N_4\rangle\langle N_5\rangle\ldots\langle N_n\rangle \\ \langle N_4\rangle\langle N_5\rangle\ldots\langle N_n\rangle \\ \langle N_5\rangle\langle N_6\rangle\ldots\langle N_n\rangle \\ \vdots \\ \vdots \\ \vdots \\ \langle N_n\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \\ p_{v_5} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_n} \end{bmatrix}$$

Here, the notations $N_4, N_5, \ldots, N_n$ are as follows:

$N_4 = \alpha$(4,1)α(4,2)α(4,3)

$N_5 = \alpha$(5,1)α(5,2)α(5,3)α(5,4)

$\vdots$

$\vdots$

$\vdots$

$N_n = \alpha$(n,1)α(n,2),. . . $\alpha(n,n-1)$

Proof We use mathematical induction to prove the algorithm. Let the number of vertices in K_n be n. For n = 3, K_3 is a cycle graph and it has already been proved in [55] that a generalized spline on K_3 is of the form:

$$p_{K_3} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3) \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3) \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \end{bmatrix}$$

As discussed before, we get the generalized spline p_{K_4} for the complete graph K_4 by adding one vertex and three edges to K_3 and the ring of generalized splines R_{K_4} will have elements of the type

$$p_{K_4} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \\ \langle\alpha(4,1)\alpha(4,2)\alpha(4,3)\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \end{bmatrix}$$

Clearly, the difference $p_{v_i} - p_{v_j}$ of adjacent vertices v_i and v_j is a multiple of α(i, j) $\in A(i,j)$, where $A(i,j)$ is the edge label for the edge joining v_i and v_j. We conclude that p_{K_4} satisfies the edge condition for generalized spline over the graph K_4.

Inductive step Assume that there exists a generalized spline $p_{K_{n-1}}$ for the complete graph K_{n-1} . Then we have generalized spline $p_{K_{n-1}}$ defined as

$$p_{K_{n-1}} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle N_4\rangle\langle N_5\rangle\ldots\langle N_{n-1}\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle N_4\rangle\langle N_5\rangle\ldots\langle N_{n-1}\rangle \\ \langle N_4\rangle\langle N_5\rangle\ldots\langle N_{n-1}\rangle \\ \langle N_5\rangle\langle N_6\rangle\ldots\langle N_{n-1}\rangle \\ \vdots \\ \vdots \\ \vdots \\ \langle N_{n-1}\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \\ p_{v_5} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_{n-1}} \end{bmatrix}$$

Where N_4, N_5,..., N_{n-1} are defined as

$N_4 = \alpha(4,1)\alpha(4,2)\alpha(4,3)$

$N_5 = \alpha(5,1)\alpha(5,2)\alpha(5,3)\alpha(5,4)$

$\vdots$

$\vdots$

$\vdots$

$N_{n-1} = \alpha(n-1,1)\alpha(n-1,2)\ldots\alpha(n-1,n-2)$

Let the vertex 'v_n' and the new edges joining the vertex v_n to the remaining $(n-1)$ vertices be added to K_{n-1} to obtain the complete graph K_n. Let the edge labels of the newly added edges be the ideals $A(n,1), A(n,2), \ldots, A(n,n-1)$ of the ring R. Taking the nth vertex label as $p_{v_n} = \alpha(n,1)\alpha(n,2)\ldots\alpha(n,n-1) = N_n$, where $\alpha(n,j) \in A(n,j)$,for $j = 1,2,\ldots,n-1$ and multiplying each vertex label of the generalized spline for K_{n-1} by N_n , we get the generalized spline p_{K_n} for K_n as:

$$p_{K_n} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(1,3)\langle N_4\rangle\langle N_5\rangle\dots\langle N_n\rangle \\ (\alpha(1,2)+\alpha(2,3))\alpha(1,3)\langle N_4\rangle\langle N_5\rangle\dots\langle N_n\rangle \\ \langle N_4\rangle\langle N_5\rangle\dots\langle N_n\rangle \\ \langle N_5\rangle\langle N_6\rangle\dots\langle N_n\rangle \\ \vdots \\ \vdots \\ \vdots \\ \langle N_n\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \\ p_{v_5} \\ \vdots \\ \vdots \\ \vdots \\ p_{v_n} \end{bmatrix}$$

Where $N_n = \alpha(n,1)\alpha(n,2)\dots\alpha(n,n-1)$ is the vertex label for the new vertex v_n. Here we can see the difference between the vertex labels of the vertices v_n and any of the remaining n - 1 vertices of K_{n-1} is a multiple of α(n, j) $\in A(n,j)$, for $j = 1,2,\dots,n-1$. Hence, we conclude that p_{K_n} satisfies the edge conditions for the generalized spline for K_n. We give software code for the above algorithm using Python. Using this we can obtain generalized spline p_{K_n} for the complete graph, K_n , for n $\geq$ 3. In this code we have used $A(i,j)$ as the notation for the ideal as well as for the elements of the ideal.

- **Python code for K_n[47]**

  The Python code is given as

```
import numpy as np
K3 = np.array(['0',"A{1,2}*A{1,3}","(A{1,2}+A{2,3}*(A{1,3})"])
def generate Kn(n):
if n leq 3 :
return K3
else:
ans = K3
for i in range (4,n+1):
j= np.hstack([ans," "])
symbol_arr = list()
a = " "
for k in range (1,i):
a = a +"A{"+str(i)+","+str(k)+"}"
ans = [ ]
for x in j:
if x! = '0':
ans.append(x+'*'+a)
else:
ans.append(x)
return ans
generate_Kn( )
```

Listing 3.1: Python code for K_n

Next we discuss the Complete Bipartite graphs.

- **Complete Bipartite graphs** K_{n_1,n_2}[47]

  Let K_{n_1,n_2} (V_1,V_2,E) be a complete bipartite graph with vertices partitioned into two disjoint sets V_1 and V_2, consisting of n_1 and n_2 vertices respectively. Let R be a commutative ring with unity which is an integral domain and let S denote the set of all non-zero ideals of R. We now extend our method to develop an algorithm for the elements of the generalized spline ring $R_{K_{n_1,n_2}}$, for the complete bipartite graph K_{n_1,n_2}. We consider the simple cases for n_1 ,n_2 = 1, 2 and 3. The vertices are ordered in the clockwise sense, starting with the first left hand side vertex in the set V_1 as the initial vertex.

  It can be easily seen that p constructed in each of the following situations is a generalized spline since the edge conditions are satisfied by the vertex labels of the adjacent vertices.

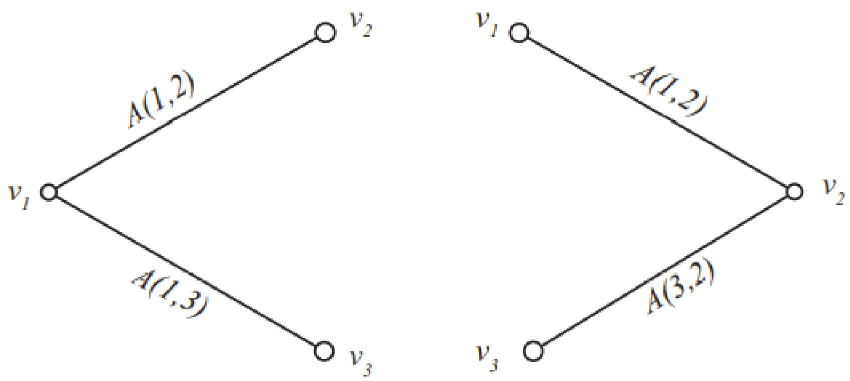

FIG. 3.5: Generalized spline on $K_{1,2}$ and $K_{2,1}$[47]

$$p_{K_{1,2}} = \begin{bmatrix} 0 \\ \alpha(1,2) \\ \alpha(1,3) \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \end{bmatrix}$$

$$p_{K_{2,1}} = \begin{bmatrix} 0 \\ \alpha(1,2)\alpha(2,3) \\ 0 \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \end{bmatrix}$$

Here the spline $p_{K_{1,2}}$ is nontrivial since α(1,2) and α(1,3) are non-zero and also the spline $p_{K_{2,1}}$ is nontrivial since R is an integral domain.

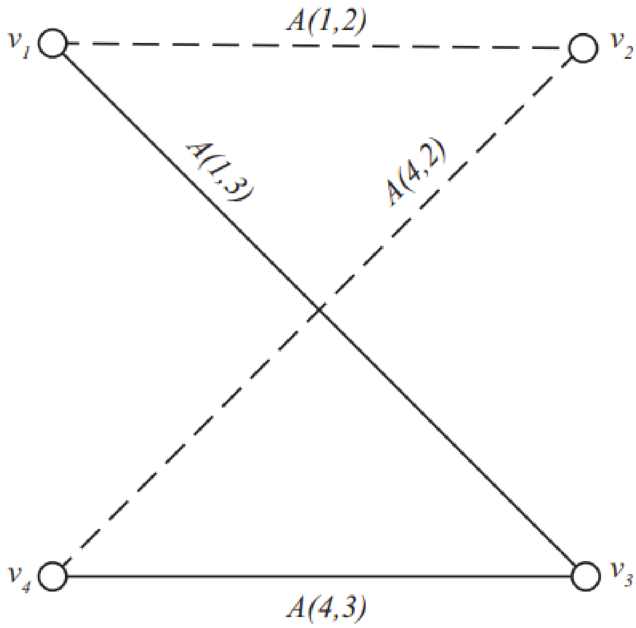

FIG. 3.6: Generalized spline on $K_{2,2}$[47]

Next we consider complete bipartite graph $K_{2,2}$. With the clockwise ordering of the vertices, we have the generalized spline for the complete bipartite graph $K_{2,2}$ as given below.

$$p_{K_{2,2}} = \begin{bmatrix} 0 \\ \alpha(1,2)\langle\alpha(4,2)\alpha(4,3)\rangle \\ \alpha(1,3)\langle\alpha(4,2)\alpha(4,3)\rangle \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \end{bmatrix}$$

Since $K_{1,2}$ or $K_{2,1}$ is a sub graph of $K_{2,2}$ and $R_{K_{1,2}}$, $R_{K_{2,1}}$ contain nontrivial generalized splines, $R_{K_{2,2}}$ also contains nontrivial generalized splines. It can be easily seen that the edge conditions are satisfied by the vertex labels of the adjacent vertices.

Also if we consider complete bipartite graph $K_{3,3}$ as in Fig. 3.7, we have the generalized splines as follows

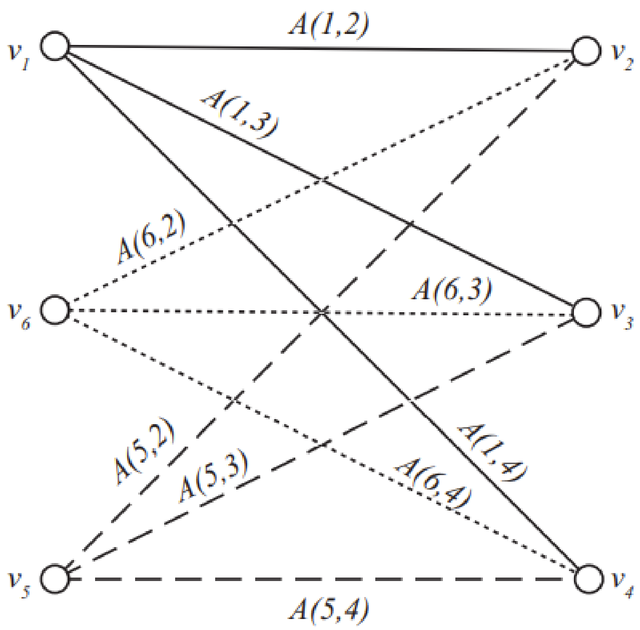

FIG. 3.7: Generalized spline on $K_{3,3}$[47]

$$p_{K_{3,3}} = \begin{bmatrix} 0 \\ \alpha(1,2)\langle N_5\rangle\langle N_6\rangle \\ \alpha(1,3\langle N_5\rangle\langle N_6\rangle \\ \alpha(1,4)\langle N_5\rangle\langle N_6\rangle \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_4} \\ p_{v_5} \\ p_{v_6} \end{bmatrix}$$

We define N_5 and N_6 as $N_5 = \alpha(5,2)\ \alpha(5,3)\ \alpha(5,4)$, $N_6 = \alpha(6,2)\ \alpha(6,3)\ \alpha(6,4)$.

Next, we consider the general case of complete bipartite graph, where the vertex sets V_1 and V_2 contain n_1 and n_2 vertices respectively. Here we introduce the notation $N_{n_2+i} = \alpha(\text{n2 + i, 2})\ \alpha(n_2 + \text{i}, 3)\ .\ .\ .\ \alpha(n_2 + \text{i}, n_2 + 1)$ for $i = 2, 3, \ldots n_1$.The following theorem gives the algorithm for writing the generalized spline ring $R_{K_{n_1,n_2}}$

- **Theorem**[47]

Let K_{n_1,n_2} be a complete bipartite graph with vertices partitioned into two disjoint sets V_1 and V_2, consisting of n_1 and n_2 vertices respectively(Fig.3.8). Then, ordering the vertices in clockwise sense as before, the following $p_{K_{n_1,n_2}}$ gives a generalized spline for the complete bipartite graph K_{n_1,n_2} .

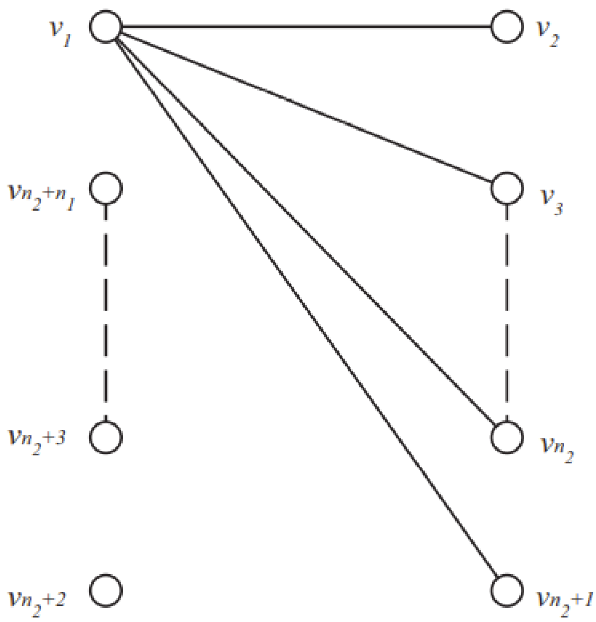

FIG. 3.8: Generalized spline on K_{n_1,n_2}

$$p_{K_{n_1,n_2}} = \begin{bmatrix} 0 \\ \alpha(1,2)\langle N_{n_2+2}\rangle\langle N_{n_2+3}\rangle\ldots\langle N_{n_2+n_1}\rangle \\ \alpha(1,3)\langle N_{n_2+2}\rangle\langle N_{n_2+3}\rangle\ldots\langle N_{n_2+n_1}\rangle \\ \vdots \\ \vdots \\ \alpha(1,n_2+1)\langle N_{n_2+2}\rangle\langle N_{n_2+3}\rangle\ldots\langle N_{n_2+n_1}\rangle \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ \vdots \\ \vdots \\ p_{v_{n_2+1}} \\ \vdots \\ \vdots \\ p_{v_{n_2+n_1}} \end{bmatrix}$$

where

$N_{n_2+i} = \alpha(n_2+i,2)\alpha(n_2+i,3)\ldots\alpha(n_2+i,n_2+1)$,for $i = 2,3,\ldots,n_1$.

Proof The proof of the above theorem follows from the observation that the difference of the vertex labels of adjacent vertices is a multiple of the elements belonging to the corresponding edge ideals. However, we note that the algorithm for generating a generalized spline for any complete bipartite graph holds only for the particular ordering of the vertices in the clockwise sense. Here $N_{n_2+i} = \alpha(n_2+i,2)\alpha(n_2+i,3)\ldots\alpha(n_2+i,n_2+1)$,for $i = 2,3,\ldots,n_1$ is non-zero since R is an integral domain. Also since K_{n_1-1,n_2-1} is sub graph of K_{n_1-1,n_2-1} and $R_{K_{n_1-1,n_2-1}}$ contains non-trivial generalized splines, $R_{K_{n_1,n_2}}$ also contains nontrivial generalized splines. Here we give the software code for the above algorithm using Python. Using this we can obtain generalized spline $p_{K_{n_1,n_2}}$ for K_{n_1,n_2}, for any value of n_1, n_2. We have used the notation $A(i,j)$ for the ideal as well as for the elements of the ideal.

- Python code for K_{n_1,n_2}[47]

```
import numpy as np
n1 = int(input('Enter n1'))
n2 = int(input('Enter n2'))
L1 = []
for i in range (0,n1+n2,1):
if i<n2+1:
if i==0:
L1.append(str(0))
L1= np.array(l1)
L1 = L1.reshape(-1,1)
Enter n1
Enter n2
RL = []
L = []
for i in range(2,n1+1):
for j in range (0,n2,1):
L.append("A{"+str(n2+i)+","+str(j+2)+"}")
RL.append(L)
L=[]
print (L1,'*',RL)
```

Listing 3.2: Python code for K_{n_1,n_2}

In the upcoming subsection, we give the method of writing the generalized spline for the n-dimensional hypercube Q_n.First we discuss about the hypercube and its general properties.

- **Hypercubes**[47]

Before constructing the generalized splines for the n-dimensional hypercube Q_n, we discuss about the Gray code[84] , which was given by Frank Gray in 1947 to prevent the spurious output from electro-chemical switches. In the present time, they are widely used for error correction in digital communications. The Gray code is an n-bit code which is an ordering of the 2^n strings of length n over 0, 1, such that every pair of successive strings differ in exactly one position. For example a 2-bit Gray code is 00, 01, 11, 10 and a 3-bit Gray code is 000, 001, 101, 111, 011, 010,110, 100. These Gray codes exists for all n [35]. Here we discuss about the n-dimensional hypercube Q_n, which is a regular graph with 2^n vertices, where each vertex corresponds to a binary string of length n [35]. Two vertices labeled by strings x and y are joined by an edge if x can be obtained from y by changing a single bit. The hypercube for n = 1,2,3 are shown in Fig.3.9.

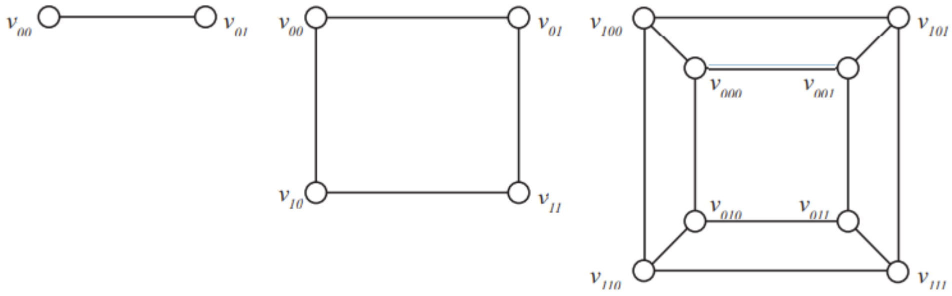

FIG. 3.9: Hypercubes Q_1, Q_2 and Q_3[47]

Interestingly, the existence of one dimensional Gray code is related to a basic property of the n-dimensional hypercube Q_n, which says that for every integer $n \geq 2$, Q_n has a Hamiltonian cycle[49]. Here, the term Hamiltonian cycle means a cycle in a graph G that contains all the vertices exactly once in G . The following Fig.3.10 expresses the Hamiltonian property of Q_2 and Q_3.

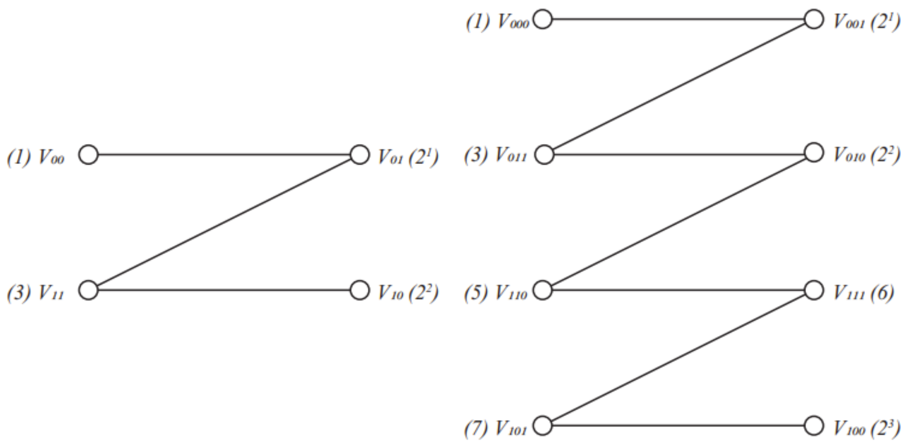

FIG. 3.10: Hamiltonicity of Hypercubes Q_2 and Q_3[47]

We define an ordering of the vertices of the hypercube in the same way as they appear in the Hamiltonian cycle. Thus, we number the vertices $1, 2, 3, \ldots, 2^n$ as shown in Fig.3.10, with the vertices $2, 4, 8, \ldots$ expressed as $2, 2^2, 2^3, \ldots, 2^n$ and call this the Hamiltonian ordering[49]. This helps us in identifying pattern in which the non-zero vertex labels appear in the generalized spline for the n-dimensional hypercube. Also, hypercubes are regular graphs with degree of each vertex equal to n. Another important property of hypercubes which we have used in the construction of generalized splines is the bipartite nature of these graphs[52]. This means that the vertex set of hypercube can be partitioned into two subsets V_1 and V_2 such that

1. No vertices of either of the subsets V_1 and V_2 are adjacent to vertices within the same set.
2. Every vertex in V_1 is adjacent to exactly n vertices V_2 and vice versa.

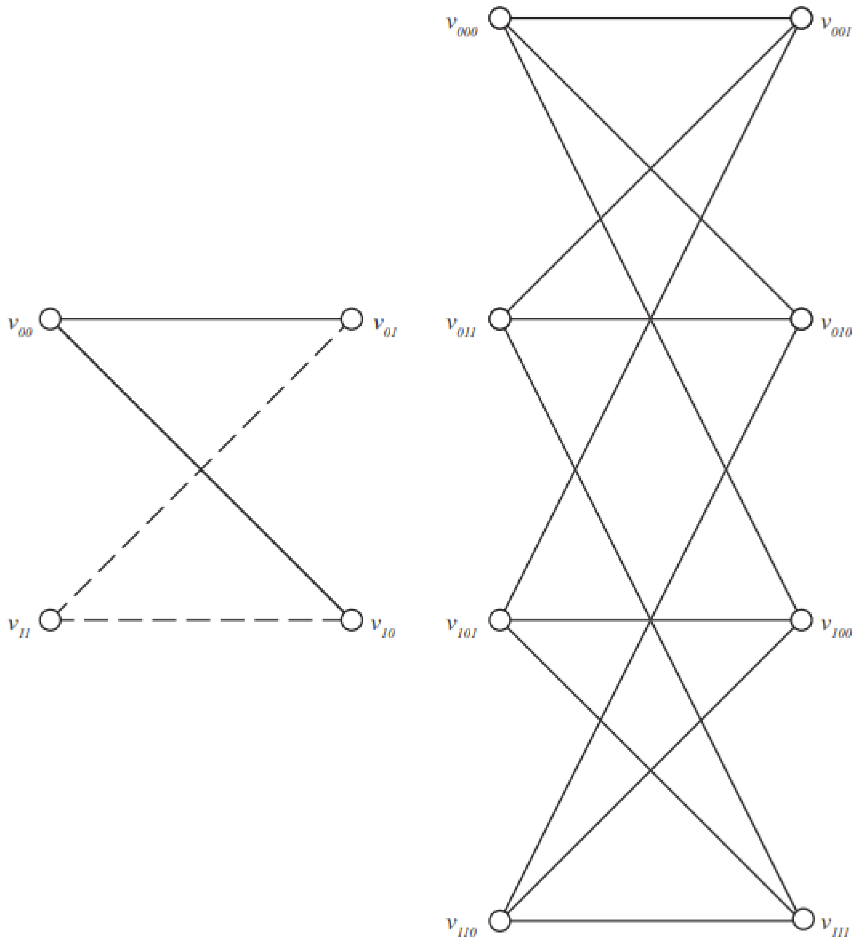

FIG. 3.11: Bipartite structure of Hypercubes Q_2 and Q_3[47]

The bipartite representation of the hypercubes for n = 2 and n = 3 are shown in Fig.3.11.Before we give the algorithm for finding the elements of R_{Q_n} for any $n > 0$, we first discuss the cases for n = 2 and 3.

- **Generalized spline for the hypercube** Q_2[47]

In this section we construct generalized spline for the graph Q_2 over R which is a commutative ring with identity and also an integral domain. The edges of Q_2 are labeled with non-zero ideals of R. The vertices are ordered in the way they appear in Hamiltonian cycle(Fig.3.10).

Then it can be easily verified that a generalized spline for Q_2 is given by

$$p_{Q_2} = \begin{bmatrix} 0 \\ \alpha_{01,00}\alpha_{01,11} \\ \alpha_{10,00}\alpha_{10,11} \\ 0 \end{bmatrix} = \begin{bmatrix} p_{v_{00}} \\ p_{v_{01}} \\ p_{v_{11}} \\ p_{v_{10}} \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_{2^2}} \end{bmatrix}$$

Here we have used similar notations in previous sections, i.e.,$\alpha_{ij,rs}$, (for i, j, r, s = 0 or 1) denote an element of the edge ideal associated with the edge joining the vertices v_{ij} and v_{rs}. Interestingly, we note that the non-zero vertex labels in p_{Q_2} appear for the vertices 2 and 2^2.

In order to construct the generalized splines for the hypercube Q_3,we refer to the bipartite structure[35] and Hamiltonian ordering[49] of Q_3(Fig.3.10 and Fig.3.11). Then it can be easily verified that a generalized spline for Q_3 is given by

$$p_{Q_3} = \begin{bmatrix} 0 \\ \alpha_{001,000}\alpha_{001,011}\alpha_{001,101} \\ 0 \\ \alpha_{010,000}\alpha_{010,011}\alpha_{010,110} \\ 0 \\ 0 \\ 0 \\ \alpha_{100,000}\alpha_{100,011}\alpha_{100,110} \end{bmatrix} = \begin{bmatrix} p_{v_{000}} \\ p_{v_{001}} \\ p_{v_{011}} \\ p_{v_{010}} \\ p_{v_{111}} \\ p_{v_{110}} \\ p_{v_{101}} \\ p_{v_{100}} \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_{2^2}} \\ p_{v_5} \\ p_{v_6} \\ p_{v_7} \\ p_{v_{2^3}} \end{bmatrix}$$

The vertices of Q_3 are $v_{i_1i_2i_3}$ where (i_1,i_2,i_3) is a binary string of length 3 and two vertices are adjacent if their respective strings differ only at one position. Also, we see that the Hamiltonian cycle in Q_3 is one in which the vertices follow a 3-bit gray code 000,001, 011, 010, 110, 111, 101, 100. We again give the Hamiltonian ordering[49] to the vertices in Q_3 by numbering the vertices 000,. . .,100 as 1,2,. . .,8. Constructing the generalized spline for Q_3 starts with labeling the vertex v_{000} as 0. Now, the vertices adjacent to v_{000} are v_{100}, v_{010} and v_{001} ,which are numbered as 2,2^2,2^3 according to Hamiltonian ordering of the vertices. We see that these are the only vertices which are labeled with non-zero elements in p_{Q_3}. Also, the vertex labels of these vertices are obtained by taking the product of the elements belonging to the edge ideals corresponding to the three edges which are adjacent to these vertices. It can be verified that with these vertex labelings, p_{Q_3} becomes a generalized spline for the hypercube Q_3, because the edge conditions are satisfied by the vertex labels of adjacent vertices. We can extend the above method of writing the generalized spline to higher dimensional hypercubes.

- **Generalized spline for the hypercube Q_4[47]**

  The graph of 4-dimensional hypercube Q_4 is in the following figure, Fig.3.12.

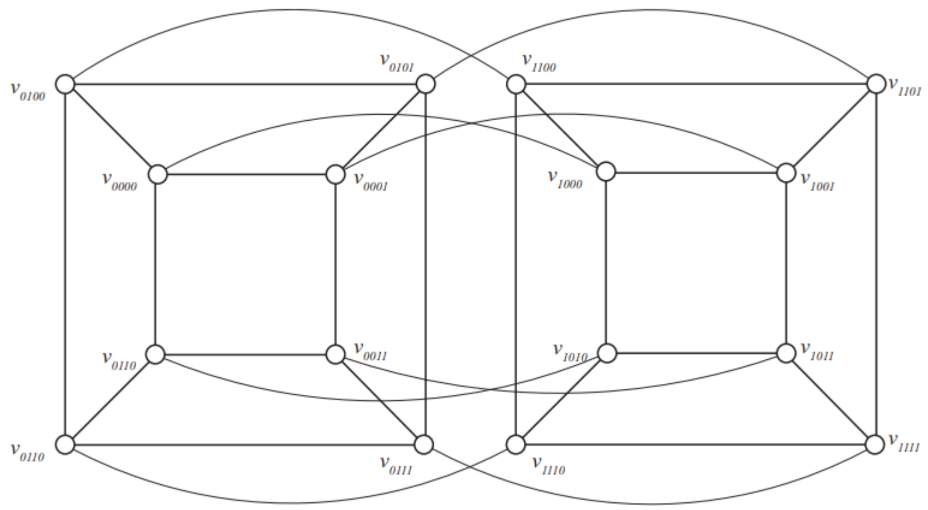

FIG. 3.12: Graph of Hypercube Q_4[47]

The bipartite structure and Hamiltonian path of the hypercube Q_4 are as follows

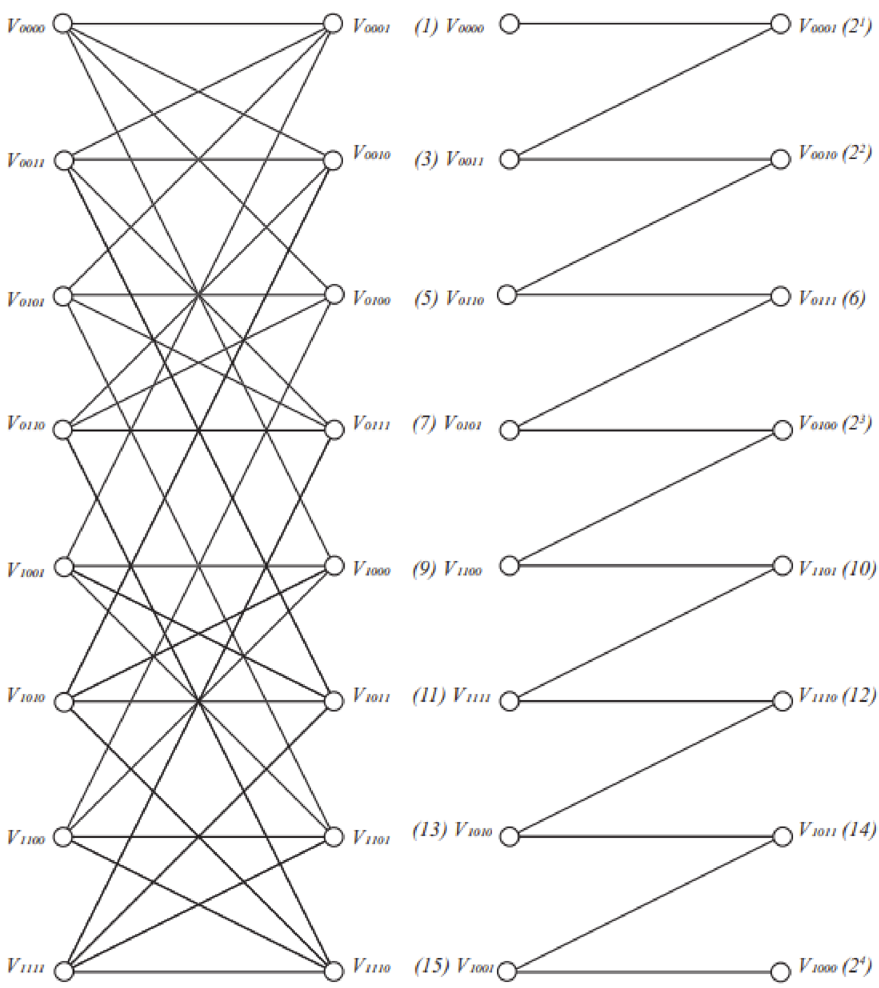

FIG. 3.13: Bipartite structure and Hamiltonicity of Hypercube Q_4[47]

For Q_4 we have the first vertex as v_{0000} which is adjacent to the vertices v_{0001},v_{0010}, v_{0100} and v_{1000}. Using the bipartite structure of Q_4 and Hamiltonian ordering, we get the generalized spline for Q_4 as follows

$$p_{Q_4} = \begin{bmatrix} 0 \\ \alpha_{0001,0000}\alpha_{0001,1001}\alpha_{0001,1010}\alpha_{0100,1100} \\ 0 \\ \alpha_{0010,0000}\alpha_{0010,1001}\alpha_{0010,1010}\alpha_{0100,1100} \\ 0 \\ 0 \\ 0 \\ \alpha_{0100,0000}\alpha_{0100,0011}\alpha_{0100,1010}\alpha_{0100,1100} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha_{1000,0000}\alpha_{1000,1001}\alpha_{1000,1010}\alpha_{1000,1100} \end{bmatrix} = \begin{bmatrix} p_{v_{0000}} \\ p_{v_{0001}} \\ p_{v_{0011}} \\ p_{v_{0010}} \\ p_{v_{0111}} \\ p_{v_{0110}} \\ p_{v_{0101}} \\ p_{v_{0100}} \\ p_{v_{1100}} \\ p_{v_{1101}} \\ p_{v_{1111}} \\ p_{v_{1110}} \\ p_{v_{1010}} \\ p_{v_{1011}} \\ p_{v_{1001}} \\ p_{v_{1000}} \end{bmatrix} = \begin{bmatrix} p_{v_1} \\ p_{v_2} \\ p_{v_3} \\ p_{v_{2^2}} \\ p_{v_5} \\ p_{v_6} \\ p_{v_7} \\ p_{v_{2^3}} \\ p_{v_9} \\ p_{v_{10}} \\ p_{v_{11}} \\ p_{v_{12}} \\ p_{v_{13}} \\ p_{v_{14}} \\ p_{v_{15}} \\ p_{v_{2^4}} \end{bmatrix}$$

Once again, we see that the non-zero vertex labels appear only for the vertices numbered as $2, 2^2, 2^3$ and 2^4. These are the vertices adjacent to the vertex 1 in the Hamiltonian ordering of the vertex v_{0000} in the bipartite structure. Also, the non-zero vertex labels are obtained by taking the product of the four elements of the edge ideals corresponding to the four edges which are incident to the respective vertices. Thus, the vertex v_{0001} is labeled with the product of the four elements $\alpha_{0001,0000}$ $\alpha_{0001,0011}$ $\alpha_{0001,0101}$ $\alpha_{0001,1001}$, because it is adjacent to the vertices v_{0000}, v_{0011}, v_{0101} and v_{1001}. This gives us an algorithm for writing the generalized spline for the edge labeled n-dimensional hypercube Q_n, for any n.

- **Theorem**[47]

Let Q_n be an n-regular hypercube with the vertices partitioned into two disjoint subsets V_1 and V_2, containing 2^{n-1} vertices each. We introduce the Hamiltonian ordering[49] for the vertices of Q_n so that the vertices are numbered as 1, 2, 3, $2^2, \ldots, 2^n$. Let the first vertex be $v_{00\ldots0}$ in V_1 and adjacent vertices $v_{0\ldots01}$, $v_{0\ldots010}$, $v_{0\ldots100}$,. . ., $v_{10\ldots0}$ in V_2 which are numbered as $2, 2^2, 2^3, \ldots, 2^n$. The vertex labels corresponding to the generalized spline p_{Q_n} defined for Q_n are as follows:

1. The vertex $v_{00\ldots0}$ is labeled with the element $0 \in R$, i.e, $p_{v_{0\ldots0}} = 0$.

2. The vertex $v_{0...1}$ which is adjacent to $v_{0...0}$ is labeled as $p_{v_{0...1}}$ and is equal to the product of the n elements belonging to the edge ideals associated with the n edges adjacent to $v_{00...0}$. Then,

$p_{v_{00...01}} = \alpha_{0...01,0...00}\alpha_{00...01,0...011}\alpha_{00...01,0...0101}\cdots\alpha_{00...01,10...01}$

Similarly the vertex $v_{00...10}$ is labeled as $p_{v_{0...10}}$ associated with the n edges adjacent to the vertex $v_{00...010}$. Then,

$p_{v_{00...010}} = \alpha_{0...10,0...00}\alpha_{00...10,0...011}\alpha_{00...10,0...0110...}\alpha_{00...10,10...010}$ and so on.

These are the only vertices with non-zero vertex labels where each vertex label is a product of n elements belonging to n edge ideals and the remaining vertices are labeled as zero. It can be easily verified that p_{Q_n} is a generalized spline on the hypercube Q_n as the edge conditions are satisfied for the adjacent vertices and also, p_{Q_n} is nontrivial since R is an integral domain.

3.4 Conclusions

Conclusions[47]

We conclude our work by developing an algorithm to construct the generalized spline rings for the special graphs such as the complete graphs, complete bipartite graphs and hypercubes. These graphs find important applications in network and approximation theory and the present work adds to the existing knowledge and understanding in these and related areas. Also, it opens a vast field for research as we can think of studying the generalized splines over these and other graphs by changing the base rings to other rings such as the polynomial rings and ring of Laurent polynomials. As these rings are PIDs, we can also try to find suitable bases for the generalized splines for these graphs.