Chapter 4

Module Basis for Generalized Spline Modules

4.1 Introduction

As discussed in chapter 2, the generalized splines were defined by Gilbert,Shira Polstar,Juliana Tymoczko in [55], over an edge labeled graph (G, α), with the base ring R and the edge labeling function α. They have shown that the set of generalized splines has a ring structure and an R-module structure with respect to pointwise operations of addition, multiplication and multiplication by elements in R respectively (section 2.4). The problem that has been addressed in the study of generalized spline modules is to determine whether these modules are free, and characterization of the bases elements in case they are free. It has been shown in [55] that for R to be a PID, the generalized spline module $R_{(G,\alpha)}$ is free over any arbitrary graph G. In fact, if G is a tree graph, $R_{(G,\alpha)}$ is free irrespective of the choice of the base ring. Flow- up classes were introduced by Handcshy, Melnick,in [63],for the cycle graphs and it was shown that the leading non zero entries of the flow-up classes were crucial in determining whether these classes formed a basis. Selma Altinok, Samet Sarioglan [7],[8] have used combinatorial techniques to determine the leading entries of the flow-up classes, using the zero trail methods which helps in calculating a crucial element Q_G in R for some graphs$G$ such as the cycle graph,tree graph etc. Basis criteria for a set of splines to become a basis for $R_{(G,\alpha)}$ is given, using the element Q_G. In this chapter, we have given the basis criteria for a set of splines to become a basis for the family of graphs such as the Dutch Windmill graph and the special cases such as the butterfly graph and friendship graph over a GCD domain R. The Dutch windmill graph consists of m copies of cycle graphs C_n, connected at a common cut vertex.

We have also extended the results to the complete graph K_4 and the wheel graph W_4, which are isomorphic and have concluded that they have the same basis criteria over a GCD domain.

4.2 Preliminaries

In this section, we give the preliminary results that are being used in our study. Throughout this study, R stands for a commutative ring and I is the set of all ideals of R.We give the definitions and results from [8],[7] and [55], which we have used for proving results in our study.By a graph $G = (V, E)$, we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively.

We have already discussed the definition of generalized splines and flow-up basis in chapter 2. However, we again give the definitions and examples for the convenience of understanding our work.

- **Generalized Spline**[7]
  A generalized spline on an edge labeled graph (G, α) is a vertex labeling $F \in R^{|V|}$ such that for each edge $v_iv_j \in E$, we have $f_i - f_j \in \alpha(v_iv_j)$, where f_i denotes the label on vertex v_i. The collection of all generalized splines on a base ring R over the edge labeled graph (G, α) is denoted by $R_{(G,\alpha)}$.

  As an example discussed in [8], we have

- **Example**[8]
  Let (G, α) be as the Fig.4.1 below.

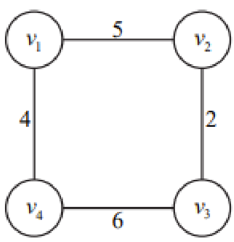

FIG. 4.1: Example of spline

A spline over (G, α) can be given by $F = (2, 12, 14, 26)$.

The flow-up classes are a special type of splines which play a very important role in determining the R-module basis for $R_{(G,\alpha)}$. They are defined as

- **Flow-up class**[7]

  Let (G, α) be an edge labeled graph with n vertices. Fix i with $1 \leq i \leq n$. A flow-up class $F^{(i)}$ is a spline in $R_{(G,\alpha)}$ with first $i-1$ leading zeros, that is, the components $F_i^{(i)} = 0$ and $F_j^{(i)} = 0$ for all $j < i$. The set of all i-th flow-up classes is denoted by F_i.

  As an example of flow-up basis, we have

- **Example**[8]

  Consider the edge labeled graph (G, α) in Fig.4.1 again. Flow-up classes on (G, α) can be given by $F^{(1)} = (1,\ 1,\ 1,\ 1)$, $F^{(2)} = (0,\ 10,\ 0,\ 0)$, $F^{(3)}(3) = (0,\ 0,\ 2,\ 0)$ and $F^{(4)} = (0,\ 0,\ 0,\ 12)$.

  The following definition of $n \times k$ matrix shows the matrix representation of a flow-up basis.

- **Definition** [8]

  Let (G, α) be an edge labeled graph with
  V(G) $=\{v_1,\ v_2,\ \ldots,\ v_n\ \}$. Let A $= \{F_1,\ F_2,\ \ldots, F_k\} \subset R_{(G,\alpha)}$.
  Let $F^i(\ v_j) \ = f_{ij}$. Then the $n \times k$ matrix

$$\begin{bmatrix} f_{1n} & f_{2n} & \ldots & f_{kn} \\ \vdots & \vdots & \ldots & \vdots \\ f_{11} & f_{21} & \ldots & f_{k1} \end{bmatrix}$$

  is the matrix representation of A.

  Bowden and others [63],[21],[20] proved that flow-up classes with smallest leading entries form a module basis for $R_{(G,\alpha)}$ where R is an integral domain.

  Selma Altinok and Samet Sarioglan introduced special trails and zero trails to determine the smallest leading entries of flow-up classes over an integral domain R.They have proved the following proposition 3.3 about zero trails in [7].

- **Proposition**[7]

  Let (G, α) be an edge labeled graph with n vertices and let $F^{(i)} = (0, \ldots, 0, f_i, \ldots, f_n)$ $\in F_i$ with $i > 1$. Let v_j be a vertex with $j \geq i$ and let $p^{(j,0)}$ be an arbitrary zero trail of v_j . Then $p^{(j,0)}$ divides f_j .

  Also they have shown the existence of flow-up bases on any graph over principal ideal domains [7]. If R is not a domain, then $R_{(G,\alpha)}$ may not have a flow-up basis even it is free.The following theorem proves the existence of flow-up bases for $R_{(G,\alpha)}$ when the base ring R is a PID.

- **Theorem**[7]

  Let (G, α) has n vertices and R be a PID. Fix v_i with $i > 1$ and assume that all vertices v_j with $j < i$ are labeled by zero. Then a flow-up class $F^{(i)}$ exists with the first nonzero entry $f_i = [p^{(i,0)}]$

  In Lemma 2.3 [7], it was shown that $R_{(G,\alpha)} \cong R_{(G',\alpha)}$ if G' is obtained by reordering the vertices of G.The following lemma shows the relation between the determinant of a basis of $R_{(G,\alpha)}$ and $R_{(G',\alpha)}$ where G' is obtained by reordering the vertices of G.

- **Lemma**[8]

  Let (G, α) be an edge labeled graph with n-vertices and let $\{F_1, \ldots, F_n\}$ forms a basis for $R_{(G,\alpha)}$. Let $\sigma \in S_n$ be a permutation and let $\sigma(G, \alpha) = (G', \alpha)$ be a vertex reordering of (G, α) as defined in Lemma 2.3[8]. If $\{G_1, \ldots, G_n\}$ is a basis for $R_{(G',\alpha)}$, then $\mid F_1F_2 \ldots F_n \mid = r \mid G_1G_2 \ldots G_n \mid$ where $r \in R$ is a unit.

  The set $R_{(G,\alpha)}$ of generalized splines is a ring and an R-module [55]. The module of generalized splines $R_{(G,\alpha)}$ contains a free sub module of rank at least the number of vertices [55], and over a PID it is always free with rank equal to the number of vertices [63]. But the module of generalized splines can have essentially any rank over a ring with zero divisors [21].

  The following example shows that with two different edge labelings of cycle graph C_3, $\dim(R_{(C_3,\alpha_1)}) \neq \dim(R_{(C_3,\alpha_2)})$,where α_1,α_2 are the edge labeling functions with the base ring $\mathbb{Z}$ and the quotient ring $\mathbb{Z}/m\mathbb{Z}$ respectively.We know that $\mathbb{Z}$ is an integral domain where as $\mathbb{Z}/m\mathbb{Z}$ is not an integral domain.

- **Example**[48]

  Let E = (2,5,3) be the set of edge labels on cycle graph C_3 (Fig.4.2), α_1 :E$\longrightarrow I_1$ where I_1 denote the set of ideals of the ring of integers $\mathbb{Z}$ and α_2: $E \longrightarrow I_2$ where I_2 denote the set of ideals of quotient ring $\mathbb{Z}/15\mathbb{Z}$, the ring with zero divisors.It can be seen that the set $R_{(C_3,\alpha_1)}$ of generalized splines over $\mathbb{Z}$ is generated by {(1,1,1),(0,2,12) (0,0,15)} and so the $\dim(R_{(C_3,\alpha_1)}) = 3$. The set of generalized splines over $\mathbb{Z}/15\mathbb{Z}$ is generated by {(1,1,1),(0,2,12)} and so the $\dim(R_{(C_3,\alpha_2)}) = 2$.

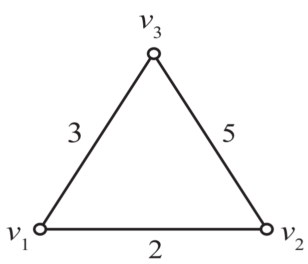

FIG. 4.2: Cycle graph C_3[48]

Selma Altinok and Samet Sarioglan[8] have defined the element $Q_G \in R$ using the method of zero trails, about which we have discussed in Chapter 2. By the notations in [8],

$Q_G = \prod_{i=2}^{k}$ [{($p_t^{(i,0)}$) } | t = 1,...,m_i] ,where $p_t^{(i,0)}$ are zero trails of v_i and m_i is the number of the zero trails of v_i.

The element Q_G could be obtained in terms of edge labels on cycles, diamond graphs and trees but not for bigger graphs in general. Gjoni[50] and Mahdavi [74] studied integer splines on cycles and diamond graphs respectively and they stated that a given set of splines forms a basis for $\mathbb{Z}_{(G,\alpha)}$ if and only if the determinant of the matrix whose columns are the elements of the given set is equal to a formula Q given by edge labels. Altinok and Sarioglan have shown in [8] that the formula Q corresponds to Q_G and generalize their statement to other families of graphs.Gjoni[50] gave basis criteria for integer splines on cycles by using determinantal techniques. The result given by Gjoni is

- **Theorem**[50]

  Fix the edge labels on (C_n, α).Let $Q = \dfrac{l_1 l_2 \dots l_n}{(l_1, l_2, \dots, l_n)}$ and let $F_1, \dots, F_n \in \mathbb{Z}_{C_n,\alpha}$.Then $\{F_1, \dots, F_n\}$ forms a module basis for $\mathbb{Z}_{C_n,\alpha}$ if and only if $\mid F_1 F_2 \dots F_n \mid = \pm Q$.

In fact, Gjoni has used the concept of flow-up basis for proving the above theorem. However, as discussed earlier, flow-up basis may not exist when R is not a PID. Altinok and Sarioglan has obtained a generalized version of Gjoni's result[50] over a GCD domain. They have shown that for a cycle graph C_n, Q defined by Gjoni is same as Q_G [8]defined by them.

Mahdavi[74] and Rose tried to give a basis criteria and obtained a similar result for the $\mathbb{Z}_{(D_{3,3},\alpha)}$ for the diamond graph $D_{3,3}$. They have given the result

- **Lemma**[74] Fix the edges on $(D_{3,3}, \alpha)$. Let $(l_2, l_3, l_4, l_5) = (l_1, l_2) = (l_1, l_3) = (l_1, l_4) = (l_1, l_5) = 1$, and $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{(l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5)}$.If $W, X, Y, Z \in \mathbb{Z}_{(D_{3,3},\alpha)}$, then Q divides $\mid W, X, Y, Z \mid$.

And they have conjectured the following result

- **Conjecture**[74]

  Fix the edges on $(D_{3,3}, \alpha)$.Let $Q = \dfrac{l_1 l_2 l_3 l_4 l_5}{(l_2, l_3)(l_4, l_5), l_1(l_2, l_3, l_4, l_5)}$ and let $W, X, Y, Z \in \mathbb{Z}_{(D_{3,3},\alpha)}$.If $\mid W, X, Y, Z \mid = \pm Q$,then $\{W, X, Y, Z\}$ forms a basis for $\mathbb{Z}_{(D_{3,3},\alpha)}$.

Again, Altinok and Sarioglan [8] have proved that that Q defined above is same as $Q_{(D_{3,3})}$ for the diamond graph and proved the conjecture made by Mahdavi and

Rose[74] for a GCD domain R, further generalizing it to the diamond graphs $D_{(m,n)}$ for any m and n. In fact, as flow-up basis is not guaranteed over a GCD domain, they have used different approach and obtained the result

- **Theorem**[8]
  Let $(D_{3,3}, \alpha)$ be an edge labeled diamond graph. Then $\{F_1, F_2, F_3, F_4\} \subset R_{(D_{3,3},\alpha)}$ is a basis for $R_{(D_{3,3},\alpha)}$ if and only if $\mid F_1F_2F_3F_4 \mid = r \cdot Q_{D_{3,3}}$ where $r \in R$ is a unit.

  Further they have generalized the result for $D_{(m,n)}$ for any m, n. Also, it is known that spline modules over tree graphs are free and possess flow-up basis irrespective of the base ring R. they have formularized Q_G for the tree graphs, and gave the basis criteria for the tree graphs using the flow-up basis. The result proved by them is

- **Theorem** [8]
  Let G be a tree with n vertices and k edges. Then $\{F_1, \ldots, F_n\} \subset R_{(G,\alpha)}$ forms a basis for $R_{(G,\alpha)}$ if and only if $\mid F_1F_2 \ldots F_n \mid = r \cdot Q_G$ where $r \in R$ is a unit and R is a GCD domain.

  The result which is very important to our work in this chapter is the one in which they have given the basis criteria over graphs which are joins of cycle, diamond and tree graphs. The result follows as

- **Corollary** [8]
  Let $\{G_1, \ldots, G_k\}$ be a collection of cycles, diamond graphs and trees and let G be a graph obtained by joining $G_1, \ldots, G_k$ together along common vertices which are cut vertices in G. Then $\{F_1,...,F_n\} \subset R_{(G,\alpha)}$ forms a basis for $R_{(G,\alpha)}$ if and only if $\mid F_1F_2 \ldots F_n \mid = r \cdot Q_{G_1} \ldots Q_{G_k}$, where $r \in$ R is a unit.

  Over a PID R, basis criteria can be obtained for an arbitrary graph G, in terms of the element Q_G. As it is difficult to calculate Q_G for complicated graphs, they have conjectured that the above results can be generalized for arbitrary graphs over GCD domains.

  In this chapter we have extended their results to Dutch Windmill graphs and their special cases such as friendship graph and butterfly graphs over GCD domains by calculating $Q_{(D_3^{(m)})}$ and giving basis criteria for a set of splines in $R_{(D_3^{(m)},\alpha)}$. Although, we could not generalize the results to arbitrary graphs, but our extension covers an important graph family and opens the possibilities of further extensions to other graphs which are widely used in networks and build upon the existing knowledge.

4.3 Results & Discussions

Results & Discussions[48] Let $(D_3^{(2)}, \alpha)$ be an edge labeled Butterfly graph (Fig. 4.3(a)) and $(D_3^{(m)}, \alpha)$ be an edge labeled Friendship Graph (Fig.4.3(b)) which are special cases of Dutch windmill graph (Fig. 4.4). An edge labeled Butterfly graph has 5 vertices v_1, v_2, v_3, v_4, v_5 and 6 edges l_1, l_2, l_3, l_4, l_5, l_6. Let v_1 be common cut vertex between two triangles T_1 and T_2.

By Corollary 3.27[8], flow-up basis for Butterfly graph over any GCD domain exists, as it has common cut vertex between two triangles (cycle graphs with 3 vertices). Thus, for any $\{F_1, F_2, F_3, F_4, F_5\} \subset R_{(D_3^{(2)}, \alpha)}$ where $F_1 = (1, 1, 1, 1, 1), F_2 = (0, g_2, g_3, g_4, g_5), F_3 = (0, 0, g_3, g_4, g_5), F_4 = (0, 0, 0, g_4, g_5)$ and $F_5 = (0, 0, 0, 0, g_5)$, we can construct the smallest leading entries of these classes using the zero trail method which is as follows

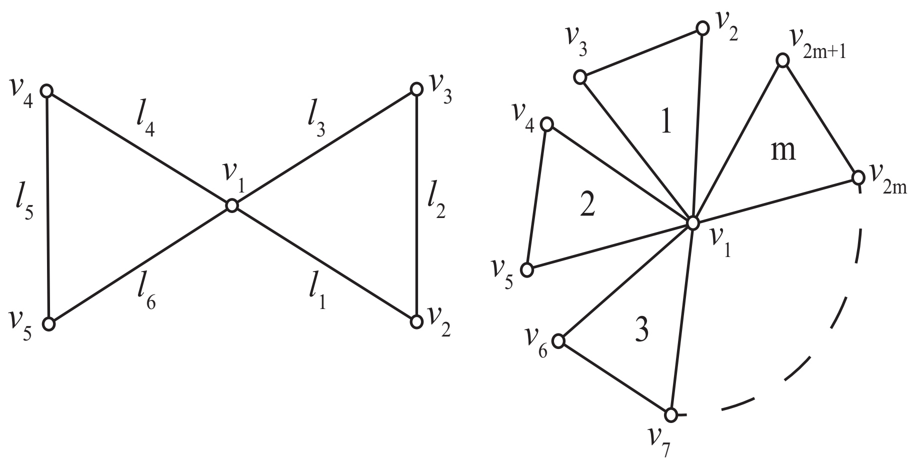

Fig. 4.3: (a) Butterfly Graph $D_3^{(2)}$ (b) Friendship Graph $D_3^{(m)}$[48]

Since zero trails of v_2 in $D_3^{(2)}$ are $p^{(2,0)} = l_1$ and $p^{(2,0)}{=}l_2\ l_3$, the smallest value for leading entry g_2 of F_2 is $[l_1, (l_2, l_3)]$ by Proposition 3.3 [7], where (l_2, l_3) denotes the greatest common divisor of edge labels l_2,l_3 and $[l_1, (l_2, l_3)]$ denotes the least common multiple of l_1 and (l_2, l_3).

Similarly, the smallest value for leading entry g_3 in F_3 is $[l_2, l_3]$, g_4 of F_4 is $[\ l_4, (l_5, l_6)\ ]$ and g_5 of F_5 is $[\ l_5, l_6\ ]$

From the definition of Q_G[8], for any graph G, we have $Q_{D_3^{(2)}}$ defined as

$$Q_{D_3^{(2)}} = [l_1,(l_2,l_3)].[l_2,l_3].[l_4,(l_5,l_6)].[l_5,l_6]$$
$$= \frac{l_1(l_2,l_3)}{(l_1,(l_2,l_3))}.\frac{l_2l_3}{(l_2,l_3)}.\frac{l_4(l_5,l_6)}{(l_4,(l_5,l_6))}.\frac{l_5l_6}{(l_5,l_6)}$$
$$= \frac{l_1l_2l_3}{(l_1,l_2,l_3)}.\frac{l_4l_5l_6}{(l_4,l_5,l_6)} = Q_{T_1}.Q_{T_2}$$

Here T_1 and T_2 are the two triangles with common cut vertex. Next we give condition for basis criterion for $D_3^{(2)}$.

- **Theorem**[48]
  Let ($D_3^{(2)}$,α) be an edge labeled Butterfly graph over any GCD domain R. Then,

  (i) Dimension of ($D_3^{(2)}$,α) = 5.
  (ii) If F = $\{F_1, F_2, F_3, F_4, F_5\} \subset R_{(D_3^{(2)},\alpha)}$ and $\mid F \mid = \mid F_1F_2F_3F_4F_5 \mid$, then F is a basis for $R_{(D_3^{(2)},\alpha)}$ if and only if
  $\mid F \mid = r.Q_{T_1}Q_{T_2}$ where r ∈ R is a unit where T_1 and T_2 are the two triangles with common cut vertex.

  Proof Taking into consideration the leading entries of F_1,F_2, F_3, F_4, F_5, as calculated earlier we have,

$$\mid F_1F_2F_3F_4F_5 \mid = \begin{vmatrix} 1 & g_5 & g_5 & g_5 & g_5 \\ 1 & g_4 & g_4 & g_4 & 0 \\ 1 & g_3 & g_3 & 0 & 0 \\ 1 & g_2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix}$$
$$= 1.g_2.g_3.g_4.g_5$$
$$= 1.[l_1,(l_2,l_3)].[l_2,l_3].[l_4,(l_5,l_6)].[l_5,l_6]$$

  Here we observe that | F | is equal to $Q_{D_3^{(2)}}$ and is equal to $Q_{T_1}.Q_{T_2}$.

  In the next lemma, we apply the above result for Friendship graph $D_3^{(m)}$ (Fig.4.3(b)).

- **Lemma**
  Let ($D_3^{(m)}$, α) be an edge labeled Friendship graph with
  2m + 1 vertices $v_1, v_2, \ldots, v_{2m+1}$ and 3m edge labels $l_1, \ldots, l_{3m}$
  (Fig.4.3(b)). It is obtained by joining m copies of triangles, $T_1, T_2, \ldots, T_m$ together along the common vertex v_1, which is cut vertex in $D_3^{(m)}$.
  Then,

$$Q_{D_3^{(m)}} = \frac{l_1 l_2 l_3}{(l_1, l_2, l_3)} . \frac{l_4 l_5 l_6}{(l_4, l_5, l_6)} \cdots \frac{l_{3m-2} l_{3m-1} l_{3m}}{(l_{3m-2}, l_{3m-1}, l_{3m})}$$

Proof The above equality can easily be shown as in [8], by rewriting the least common multiples and greatest common divisors explicitly.

Next we apply this result to Dutch windmill graph $D_n^{(m)}$, for any n and m (Fig.4.4).

In order to prove the basis criterion, first we give the formula for $Q_{D_n^{(m)}}$.

- **Lemma**[48]

Let $(D_n^{(m)}, \alpha)$ be an edge labeled Dutch windmill graph with $m(n-1)+1$ vertices $v_1, v_2, \ldots, v_{m(n-1)+1}$ and mn edge labels $l_1, \ldots, l_{mn}$ (Fig.4.4).
It is obtained by joining m copies of n-cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$
together along the common vertex v_1,
which is cut vertex in $D_n^{(m)}$.
Then $Q_{D_n^{(m)}}$ is obtained as

$$Q_{D_n^{(m)}} = \frac{l_1 l_2 \ldots l_n}{(l_1, l_2, \ldots, l_n)} . \frac{l_{n+1} \ldots l_{2n}}{(l_{n+1}, \ldots, l_{2n})} \cdots \frac{l_{mn-(n-1)} \ldots l_{mn}}{(l_{mn-(n-1)}, \ldots, l_{mn})}$$

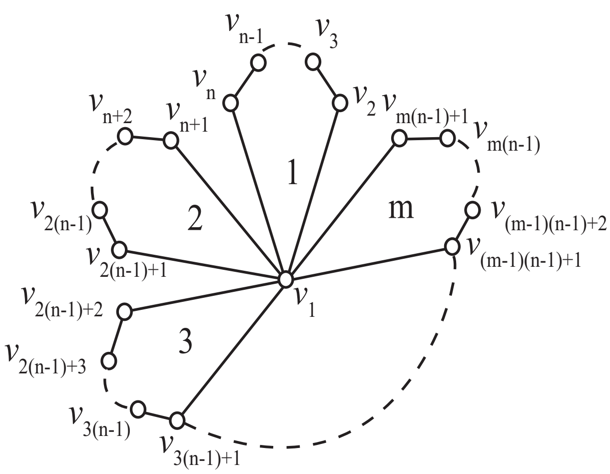

FIG. 4.4: Dutch windmill graph, $D_n^{(m)}$

Proof Again, the above equality can easily be shown as in [8], by rewriting the least common multiples and greatest common divisors explicitly.

As a special case, we can obtain the basis criteria for generalized spline modules on Friendship graph $D_3^{(m)}$ over any GCD domain as follows:

- **Theorem**

Let $(D_3^{(m)}, \alpha)$ be an edge labeled Friendship graph over any GCD domain R.Then
(i) Dimension of $(D_3^{(m)}, \alpha)$ = 2m+1.

(ii) If F $= \{F_1, \ldots, F_{2m+1}\} \subset R_{(D_3^{(m)},\alpha)}$ and $\mid F \mid = \mid F_1\ F_2 \ldots F_{2m+1} \mid$, then F is a basis for $R_{(D_3^{(m)},\alpha)}$ if and only if $\mid F \mid = r.Q_{T_1}Q_{T_2}\ldots Q_{T_m}$ where r $\in$ R is a unit where $T_1, T_2, \ldots, T_m$ are triangles with common cut vertex.

Proof First we reorder the vertices of graph $D_3^{(m)}$such as the vertices on each triangle are consecutively ordered, except the least indiced vertex v_1. Construct the matrix $[F_1\ F_2 ....F_{2m+1}]$ whose columns are elements of the obtained basis $\{F_1, \ldots, F_{2m+1}\}$ for $R_{(D_3^{(m)},\alpha)}$. Then the determinant of this matrix, $\mid F_1\ F_2 \ldots F_{2m+1} \mid$ is equal to the product r. $Q_{T_1}\ Q_{T_2} \ldots Q_{T_m}$, where r $\in$ R is a unit (By methods used in Theorem 3.8 [8] and Corollary 3.28[8]).

We can generalize the above theorem to obtain the basis criteria for generalized spline modules on Dutch windmill graph $D_n^{(m)}$ over any GCD domain as follows:

- **Theorem**[48]
Let ($D_n^{(m)}$,α) be an edge labeled Dutch windmill graph over any GCD domain R.Then
(i) Dimension of ($D_n^{(m)}$,α) $= m(n-1)+1$
(ii) If F $= \{F_1, \ldots, F_{m(n-1)+1}\} \subset R_{(D_n^{(m)},\alpha)}$ and $\mid F \mid = \mid F_1\ F_2 \ldots F_{m(n-1)+1} \mid$, then F is a basis for $R_{(D_n^{(m)},\alpha)}$ if and only if $\mid F \mid = r.Q_{C_{n_1}}Q_{C_{n_2}}\ldots Q_{C_{n_m}}$ where r $\in$ R is a unit and $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ are cycle graphs with n vertices which have common cut vertex.
Proof It follows directly from the above Lemma and Theorem for friendship graph.

Next, we consider Complete graph K_4 (Fig.4.5(a)) and Wheel graph W_4 (Fig.4.5(b)) which are isomorphic to each other. We find basis criteria for generalized spline modules on these two isomorphic graphs separately over GCD domain. First we consider basis criterion for complete graph.

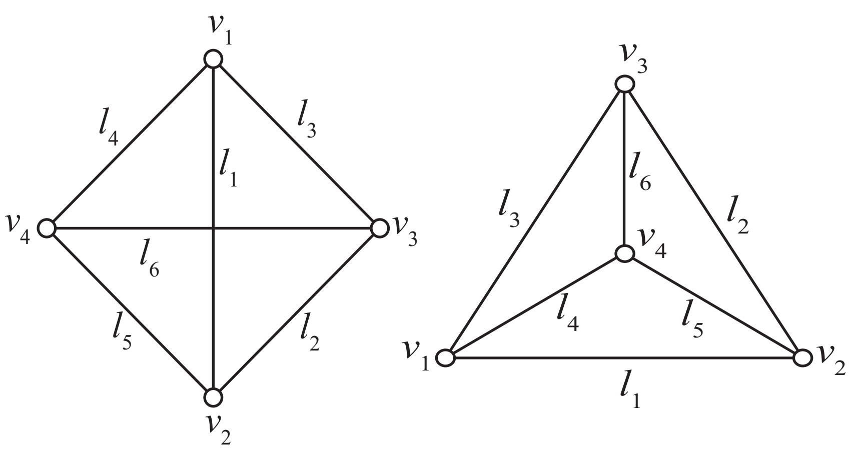

FIG. 4.5: (a) Complete Graph (b) Wheel graph W_4

- **Complete Graph** K_4[48]

  Take Diamond graph $D_{3,3}$ with the common edge l_1 and the remaining edges as l_2, l_3, l_4, l_5 and four vertices v_1, v_2, v_3, v_4. Adding one edge l_6 between the vertices v_3 and v_4, we get complete graph K_4 as above (Fig.4.5(a)). There is no common cut vertex between the Diamond graph and the edge l_6. Now, l_1, l_2, l_3, l_4, l_5 is Diamond graph with centre edge l_1 and also l_6, l_5, l_2, l_3, l_4 is a Diamond graph with centre edge l_6.

  Then the smallest leading entries of splines F_2, F_3, F_4 which are calculated by using the zero trails of vertices v_2, v_3, v_4 of Complete graph K_4 are
  $[l_1,(l_2,l_3),(l_4,l_5),(l_2,l_6,l_4),(l_5,l_6,l_3)]$; $[l_2,l_3,(l_6,l_4),(l_6,l_5)]$; $[l_4,l_6,l_5]$ respectively.

  From the definition of Q_G [8], for any graph G, we give Q_{K_4} as

$$
\begin{aligned}
Q_{K_4} &= [l_1,(l_2,l_3),(l_4,l_5),(l_2,l_6,l_4),(l_5,l_6,l_3)]\cdot[l_2,l_3,(l_6,l_4),(l_6,l_5)]\cdot[l_4,l_6,l_5]\\
&= \frac{l_1(l_2,l_3)(l_4,l_5)(l_2,l_4,l_6)(l_3,l_5,l_6)}{(l_1,(l_2,l_3),(l_4,l_5),(l_2,l_4,l_6),(l_3,l_5,l_6))}\cdot\frac{l_2l_3(l_6,l_4)(l_5,l_6)}{(l_2,l_3(l_6,l_4),(l_6,l_5))}\cdot[l_4,l_5,l_6]\\
&= \frac{(l_4,l_5)(l_5,l_6)(l_6,l_4)[l_4,l_5,l_6]l_1l_2l_3(l_2,l_3)(l_2,l_4,l_6)(l_3,l_5,l_6)}{((l_1,l_2,l_3,(l_4,l_5)),(l_2,l_4,l_6,(l_3,l_5,l_6)))(l_2,l_3,l_6,l_4,(l_6,l_5))}\\
&= \frac{l_4l_5l_6(l_4,l_5,l_6)l_1l_2l_3(l_2,l_3)(l_2,l_6,l_4)(l_5,l_6,l_3)}{((l_1,l_2,l_3,l_4,l_5),(l_2,l_4,l_6,l_3,l_5))(l_2,l_3,l_6,l_4,l_6,l_5)}\\
&\qquad [\ (l_4l_5)(l_5,l_6)(l_6,l_4)[l_4,l_6,l_5]=l_4l_5l_6(l_4,l_5,l_6)]\\
&= \frac{l_1l_2l_3l_4l_5l_6(l_2,l_3)(l_2,l_4,l_6)(l_3,l_5,l_6)(l_4,l_5,l_6)}{((l_1,l_2,l_3,l_4,l_5),(l_2,l_3,l_4,l_5,l_6))(l_2,l_3,l_4,l_5,l_6)}
\end{aligned}
$$

  In the above simplification of formula for Q_{K_4}, we have used the properties of gcd and lcm for any GCD domain. We prove following lemma to show that Q_{K_4} divides $|\ F_1\ F_2\ F_3\ F_4\ |$

  .

- **Lemma**[48]

  Let K_4 be as in Fig.4.5(a) and let$\{F_1,F_2,F_3,F_4\}\subset R_{(K_4,\alpha)}$. Then Q_{K_4} divides $|\ F_1\ F_2\ F_3\ F_4\ |$.

  Proof Since edge labels l_1, l_2, l_3, l_4 and l_5 form a Diamond graph, we conclude that $l_1l_2l_4, l_1l_2l_5, l_1l_3l_4, l_1l_3l_5, l_2l_3l_4, l_2l_3l_5, l_2l_4l_5$ and $l_3l_4l_5$ divide $|\ F_1F_2F_3F_4\ |$ by lemma3.13[8].

  Similarly, since l_6, l_5, l_2, l_3, l_4 form a Diamond graph respectively, we conclude that the products $l_6l_5l_3$,$l_6l_5l_4$, $l_6l_2l_3$, $l_6l_2l_4$, $l_5l_2l_3$, $l_5l_2l_4$, $l_5l_3l_4$ and $l_2l_3l_4$ divide $|\ F_1F_2F_3F_4\ |$, by lemma3.13 [8].

Since the products $l_6 l_2 l_4, l_6 l_5 l_3$ and $l_4 l_5 l_6$ divide $\mid F_1 F_2 F_3 F_4 \mid$, gcd of the edge labels (l_2,l_4,l_6),(l_5,l_6,l_3) and (l_4,l_5,l_6) divide $\mid F_1 F_2 F_3 F_4 \mid$.

In order to see that the product $l_2 l_3$ divides $\mid F_1 F_2 F_3 F_4 \mid$, where
$F_i = (f_{i1},\, f_{i2},\, f_{i3},\, f_{i4}) \in R_{(K_4,\alpha)}$ for i = 1,2,3,4, we consider the determinant

$$\mid F_1 F_2 F_3 F_4 \mid = \begin{vmatrix} f_{14} & f_{24} & f_{34} & f_{44} \\ f_{13} & f_{23} & f_{33} & f_{43} \\ f_{12} & f_{22} & f_{32} & f_{42} \\ f_{11} & f_{21} & f_{31} & f_{41} \end{vmatrix}$$

By some suitable row operations on the determinant and by edge conditions, we obtain

$$\mid F_1 F_2 F_3 F_4 \mid = \begin{vmatrix} f_{14} & f_{24} & f_{34} & f_{44} \\ f_{13} - f_{11} & f_{23} - f_{21} & f_{33} - f_{31} & f_{43} - f_{41} \\ f_{12} - f_{13} & f_{22} - f_{23} & f_{32} - f_{33} & f_{42} - f_{43} \\ f_{11} & f_{21} & f_{31} & f_{41} \end{vmatrix}$$

$$= \begin{vmatrix} f_{14} & f_{24} & f_{34} & f_{44} \\ x_{13} l_3 & x_{23} l_3 & x_{33} l_3 & x_{43} l_3 \\ x_{12} l_2 & x_{22} l_2 & x_{32} l_2 & x_{42} l_2 \\ f_{11} & f_{21} & f_{31} & f_{41} \end{vmatrix}$$

$$= l_2 l_3 \begin{vmatrix} f_{14} & f_{24} & f_{34} & f_{44} \\ x_{13} & x_{23} & x_{33} & x_{43} \\ x_{12} & x_{22} & x_{32} & x_{42} \\ f_{11} & f_{21} & f_{31} & f_{41} \end{vmatrix} \in R$$

for some x_{ij} belonging to R.

Hence we see that $l_2 l_3$ divides $\mid F_1\ F_2\ F_3\ F_4 \mid$ and so (l_2,l_3) divides $\mid F_1 F_2 F_3 F_4 \mid$ i.e.,

$$Q_{K_4} = \frac{l_1 l_2 l_3 l_4 l_5 l_6 (l_2, l_3)(l_2, l_4, l_6)(l_3, l_5, l_6)(l_4, l_5, l_6)}{((l_1, l_2, l_3, l_4, l_5), (l_2, l_3, l_4, l_5, l_6))(l_2, l_3, l_4, l_5, l_6)}$$

divides $\mid F_1\ F_2\ F_3\ F_4 \mid$.

Next we give following lemma for any GCD domain.

- **Lemma**[48]
  Let (K_4,α) be an edge labeled Complete graph. Let
  $\{\ F_1,\ F_2,\ F_3,\ F_4\ \} \subset R_{(K_4,\alpha)}$. If $\mid F_1\ F_2\ F_3\ F_4 \mid$ = r Q_{K_4}, where r $\in$ R is a unit, then $\{\ F_1,\ F_2,\ F_3,\ F_4\ \}$ forms a basis for $R_{(K_4,\alpha)}$.

  Proof Proof can be shown by using similar techniques as in the proof of lemma 3.17 [8].

Next, we prove following theorem to show that flow-up basis exists for generalized spline modules on Complete graph K_4 over GCD domain.

- **Theorem**

Let (K_4, α) be an edge labeled Complete graph over any GCD domain. Let $\{ F_1, F_2, F_3, F_4 \} \subset R_{(K_4,\alpha)}$. If $\{ F_1, F_2, F_3, F_4 \}$ is a basis for $R_{(K_4,\alpha)}$, then $| F_1\ F_2\ F_3\ F_4 | =$ r Q_{K_4}, where r $\in$ R is a unit.

Proof Since $\{F_1, F_2, F_3, F_4\} \subset R_{(K_4,\alpha)}$, the determinant $| F_1\ F_2\ F_3\ F_4 | =$ r Q_{K_4} for some r $\in$ R .We will show that r is a unit.

Let $d_1 = (l_2,l_3)$ and $d_2 = (l_4,l_5)$. Then we have $l_2 = d_1 l'_2$, $l_3 = d_1 l'_3$ with $(l'_2,l'_3) = 1$ and similarly $l_4 = d_2 l'_4$, $l_5 = d_2 l'_5$ with $(l'_4,l'_5) = 1$. Consider the following matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & [l_4, l_5] \\ 1 & 0 & [l_2, l_3] & 0 \\ 1 & 0 & 0 & 0 \\ 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_3 l'_5 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_4 l'_2 & 0 & [l_4, l_5] \\ 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_4 l'_2 & [l_2, l_3] & 0 \\ 1 & 0 & 0 & 0 \\ 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_4 l'_2 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_3 l'_4 & 0 & [l_4, l_5] \\ 1 & 0 & [l_2, l_3] & 0 \\ 1 & 0 & 0 & 0 \\ 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_3 l'_4 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & [l_4, l_5] \\ 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_2 l'_5 & [l_2, l_3] & 0 \\ 1 & 0 & 0 & 0 \\ 1 & [l_1, (l_2, l_3), (l_4, l_5)]l'_2 l'_5 & 0 & 0 \end{bmatrix}$$

It can be easily seen that each column of A_1, A_2, A_3 and A_4 is an element of $R_{(K_4,\alpha)}$. By proposition 3.1 [8], $| F_1\ F_2\ F_3\ F_4 | =$ r Q_{K_4} divides $l'_3 l'_5$. One can conclude that r divides $l'_2 l'_4$, $l'_3 l'_4$ and $l'_2 l'_5$ by the same observation. Thus we have,

$$\begin{aligned} r \mid (l'_3 l'_5 l'_2 l'_4 l'_3 l'_4 l'_2 l'_5) &= ((l'_2 l'_4, l'_2 l'_5), (l'_3 l'_4, l'_3 l'_5)) \\ &= (l'_2 (l'_4, l'_5), l'_3 (l'_4 . l'_5)) \qquad \text{and so } r \text{ is a unit.} \\ &= (l'_2, l'_3) = 1 \end{aligned}$$

Combining above lemma and theorem for (K_4, α), we have the following result for complete graph K_4, over any GCD domain.

- **Theorem**[48]

  Let (K_4,α) be an edge labeled Complete graph over any GCD domain. Then
  (i) Dimension of $R_{(K_4,\alpha)} = 4$.
  (ii) If F=$\{F_1,F_2,F_3,F_4\} \subset R_{(K_4,\alpha)}$ and $\mid F \mid = \mid F_1F_2\ F_3F_4 \mid$,then F is a basis for $R_{(K_4,\alpha)}$ if and only if $\mid$ F $\mid$ = rQ_{K_4}, where r $\in$ R is a unit.

  Now we find basis criterion for generalized spline modules on Wheel graph W_4 with 4 vertices, which is isomorphic to Complete graph K_4 using the above method.

- **Wheel Graph** W_4[48]

  Let W_4 be an edge labeled Wheel graph (Fig.4.5(b)). It has four vertices v_1,v_2,v_3,v_4 and 6 edges, l_1,l_2,l_3,l_4,l_5 and l_6 and is isomorphic to Complete graph K_4. The smallest leading entries of of splines F_2, F_3, F_4 which are calculated by using the zero trails of vertices v_2,v_3 and v_4 of Wheel graph W_4 are $[l_1,(l_2,l_3),(l_4,l_5),(l_2,l_6,l_4),(l_5,l_6,l_3)]$; $[l_2, l_3, (l_6,l_4),(l_6,l_5)]$ and $[l_4,l_6,l_5]$
  We observe that these leading entries are same as the smallest leading entries of splines of Complete graph K_4. From the definition of Q_G[8], for any graph G, we have
  $$Q_{W_4} = [l_1,(l_2,l_3),(l_4,l_5),(l_2,l_6,l_4),(l_5,l_6,l_3)].[l_2,l_3,(l_6,l_4),(l_6,l_5)].[l_4,l_6,l_5]$$
  $$= \frac{l_1l_2l_3l_4l_5l_6(l_4,l_5,l_6)(l_2,l_3)(l_2,l_6,l_4)(l_5,l_6,l_3)}{((l_1,l_2,l_3,l_4,l_5),(l_2,l_3,l_4,l_5,l_6))(l_2,l_3,l_4,l_5,l_6)}$$
  which is equal to Q_{K_4}

  Now, we can have following theorem for basis criterion for generalized spline modules on Wheel graph W_4 which can easily be proved with similar methods used as in case of Complete graph K_4 over any GCD domain.

- **Theorem**[48]

  Let (W_4,α) be an edge labeled Wheel graph over any GCD domain, R. Then,
  (i) Dimension of $R_{(W_4,\alpha)} = 4$.
  (ii) If F = $\{F_1, F_2, F_3, F_4\} \subset R_{(W_4,\alpha)}$ and $\mid$ F $\mid$ = $\mid F_1\ F_2\ F_3\ F_4 \mid$, then F is a basis for $R_{(W_4,\alpha)}$ if and only if $\mid$ F $\mid$ = r Q_{W_4}, where r $\in$ R is a unit.

  Here we observe that Complete graph K_4 and Wheel graph W_4 which are isomorphic to each other have same set of smallest leading entries for their flow-up splines. Also, the formula for Q_G is equal for these graphs and basis criterion for generalized spline modules on these graphs is same over any GCD domain.

4.4 Conclusions

Conclusions[48]

We have given basis criteria for $R_{(G,\alpha)}$ on edge labeled Dutch windmill graph and special cases of Dutch windmill graph such as Friendship graph and Butterfly graph which have common cut vertices with Cycle graph C_n and triangles respectively, over any GCD domain by using determinantal techniques[8] and flow-up bases. Dutch windmill graph has a lot of applications in dynamical processes like epidemic dynamics and network synchronization which are studied on graphs [51]. Our study may help in deeper understanding of structures of the graphical representation of the spread of infectious diseases which has become an active area of research in the present times.

By giving basis criteria for two graphs K_4 and W_4 over any GCD domain, we show an example for the conjecture 3.29 [8] claimed by Selma Altinok and Samet Sarioglan.We observe that graphs which are isomorphic to each other have same or equivalent basis criteria since smallest leading entries of flow-up splines of these graphs are same and thus formula, Q_G is also same for these graphs.

Further investigations on arbitrary graphs open a possibility of finding proof for general basis criteria for spline modules on arbitrary graphs over any GCD domain.