Chapter 5

Splines on Wheel Graphs over Z mod $p^k Z$

5.1 Introduction

In this chapter we extended the work done by Nealy Bowden and Juliana Tymoczko [21] on cycles C_n to wheel graphs W_{n+1}, which are an extension of the cycle graphs. Considering the base ring as the quotient ring $\mathbb{Z}/m\mathbb{Z}$, the edges of W_{n+1} are labeled by the ideals of $\mathbb{Z}/m\mathbb{Z}$, which are principal ideals. The $\mathbb{Z}$-module $R_{(W_{n+1},\alpha)}$ is finite and hence it must have a minimum generating set which functions as basis, as it may not be free. As discussed in Chapter 2, a spline module contains a free submodule of rank at least the number of vertices of the graph G over an integral domain and is free with rank equal to the number of vertices over a PID. However, over the quotient ring $\mathbb{Z}/m\mathbb{Z}$ these minimum generating sets may be smaller. In fact it has been proved in [21] that there exists a graph G with $n \geq 4$, for which the spline module over the ring $\mathbb{Z}/m\mathbb{Z}$, where m has atleast 2 prime factors, has rank k, for $k = 1, 2, \ldots, n$. They have shown that the computations of splines are reduced by using the prime factorization of m and generated an algorithm to explicitly construct a minimum generating set over cycle graphs over $\mathbb{Z}/m\mathbb{Z}$, using the flow-up spline. The structure theorem for finite abelian groups has been used for finding the conditions when the flow-up splines form the minimum generating sets. We have classified splines on wheel graphs, finding a minimum generating set of flow-up classes over $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime. We also classified splines on cycles over $\mathbb{Z}/m\mathbb{Z}$ if m has few prime factors and find a generating set of flow-up classes on these graphs over $\mathbb{Z}/m\mathbb{Z}$.

5.2 Preliminaries

In this section we give the basic results and definitions that are important for our work. The definitions of generalized splines and flow-up splines are already discussed in previous chapters. We give the definition of constant flow-up spline as given by Bowden and Tymokzho in [21].

- **Constant flow-up spline**[21]

  Flow-up spline p for which there exists an element $n_i \in \mathbb{Z}/m\mathbb{Z}$ such that $p_{v_i} \in 0, n_i$ for each $v_i \in V$.

  As an example, a set of constant flow-up splines for a graph over $\mathbb{Z}/21\mathbb{Z}$ is given below

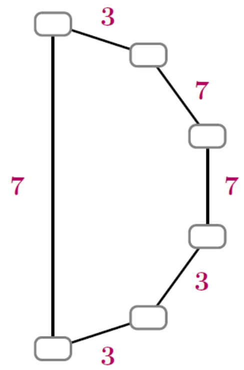

FIG. 5.1: A Graph over $\mathbb{Z}/21\mathbb{Z}$

$$\left\{ \begin{pmatrix} 1\\1\\1\\1\\1\\1 \end{pmatrix} \begin{pmatrix} 0\\3\\3\\3\\3\\0 \end{pmatrix} \begin{pmatrix} 0\\3\\3\\3\\0\\0 \end{pmatrix} \begin{pmatrix} 7\\7\\7\\0\\0\\0 \end{pmatrix} \begin{pmatrix} 7\\7\\0\\0\\0\\0 \end{pmatrix} \right\}$$

They have considered the cycle graph C_n , the base ring as $\mathbb{Z}/m\mathbb{Z}$ and labeled the cycles as shown in Fig.5.2, with the edge labels l_i as the principal ideals of the ring $\mathbb{Z}/m\mathbb{Z}$.The module $R_{(G,\alpha)}$ is finite because it is a subset of $(\mathbb{Z}/m\mathbb{Z})^n$.

The following theorem shows that $R_{(G,\alpha)}$ is generated by the flow-up splines over the base ring The following theorem shows that $R_{(G,\alpha)}$ is generated by the flow-up splines over the base ring $\mathbb{Z}/m\mathbb{Z}$.

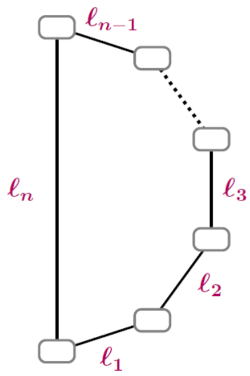

FIG. 5.2: General n-cycle with labeling

- **Theorem**

  The $\mathbb{Z}$-module R_G,is generated by a collection of flow-up splines.

  Bowden and Tymoczko [21] have shown that homomorphisms in base rings induce homomorphisms in generalized spline rings over graph G. With this they have concluded two important results for splines over $\mathbb{Z}/m\mathbb{Z}$, which are as follows
  1) The bases of spline modules over integers induce generating sets for splines over $\mathbb{Z}/m\mathbb{Z}$.

  2)The primary decomposition of the ring $\mathbb{Z}/m\mathbb{Z}$ induces decomposition of the ring of splines, i.e, that splines mod m' decompose into a direct product of splines mod m and splines mod m' when m and m' are relatively prime.

  Using these results and Theorem 5.2 which is mentioned below, Bowden and Tymoczko have given the minimum generating set for a spline module over a graph G whose edges are labeled by a single edge label $\langle a \rangle$.

- **Theorem**[21]

  If G is a connected graph such that every edge of G is with (a) where a is an element of the ring R, then a minimum generating set for R_G is

$$B(R_G) = \left\{ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a \\ 0 \end{pmatrix} \quad \cdots \quad \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

An important corollary (Corollary 5.3 in [21] to the above theorem generalizes the result over $\mathbb{Z}/p^k\mathbb{Z}$.

- **Corollary**[21]

  Let G be a graph and p be a prime number. Then splines on $\mathbb{Z}/p^2\mathbb{Z}$ are generated by the minimum generating set $B(R_G)$.

  As there exists only one nonzero, non unit edge label for G when the base ring is $\mathbb{Z}/p^2\mathbb{Z}$. This result has been extended to cycles whose edges are labeled by the powers of a as in Theorem 5.4 in [21], which is as follows

- **Theorem**[21]

  Fix a zero divisor a in $\mathbb{Z}/m\mathbb{Z}$. Suppose all of the edges of C_n are labeled with powers of a so the set of edge labels is $\{a^{k_1}, a^{k_2}, a^{k_3}, \ldots, a^{k_n}\}$. Without loss of generality assume that a^{k_1} is the minimal power in the set and that a^{k_1} is the label on edge l_n. Then the following set generates all splines on C_n.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_1 \\ \vdots \\ l_1 \\ l_1 \\ 0 \end{pmatrix} \begin{pmatrix} l_2 \\ \vdots \\ l_2 \\ l_2 \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} l_i \\ l_i \\ \vdots \\ l_i \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} l_{n-2} \\ l_{n-2} \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} l_{n-1} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

  This result has characterized the ring of splines for the cycles over the quotient ring $\mathbb{Z}/p^k\mathbb{Z}$ as in corollary 5.5 in [21].

- **Corollary**[21]

  Let C_n be the cycle on n vertices, let p be a prime number and let k be any positive integer. Then the splines on C_n over $\mathbb{Z}/p^k\mathbb{Z}$ are generated by the minimum generating set B in the above Theorem.

  We have obtained the minimum generating sets over the wheel graphs using the above results, over the quotient rings $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/p^k\mathbb{Z}$ in this chapter.

  The following proposition 3.11 is given in[21]

- **Proposition**[21]

  If G is a subgraph of G' then every spline in $R_{G'}$ restricts to a spline in R_G via the forgetful map $\pi : R_{G'} \longrightarrow R_G$ that omits the vertices in $V(G') - V(G)$ and their incident edges.

  Let G be an edge-labeled graph and let G^* be a graph obtained from G by adding a vertex v and some edges between v and G. Using the above proposition, Bowden and Tymoczko proved that each spline on the expanded graph G^* consists of the sum of a spline coming from G and a spline supported exactly on the new vertex and we have used this result in our work for wheel graph which we label as shown in [Fig.5.3].

Here we construct an edge labeled wheel graph W_{n+1} from a cycle graph v_{n+1} in the interior of the cycle C_n and corresponding vertices of C_n. This new vertex is adjacent to all the vertices of C_n. So in wheel graph the vertex labels v_1,v_2,v_3,. . ., v_n satisfy the edge conditions of v_1,v_2,. . .,v_{n+1} satisfy the edge conditions of,l_1,l_2,. . .,l_{n+1} l_{n+2},. . .,l_{2n} where l_is are shown in Fig.5.3.

The following theorems 5.1[21], 5.2[21] and corollary 5.3[21] are used for proving our results for wheel graph.

- **Theorem**[21]
  Let p be a prime number. If G is an edge-labeled graph over $\mathbb{Z}/p\mathbb{Z}$ with no edges labeled zero then every vertex-labeling over $\mathbb{Z}/p\mathbb{Z}$ is a spline on G.

- **Theorem**[21]
  If G is a connected graph such that every edge of G is with (a) where a is an element of the ring R, then a minimum generating set for R_G is

$$B(R_G) = \left\{ \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a \\ 0 \end{pmatrix} \cdots \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

- **Corollary**[21]
  Let G be a graph and p be a prime number. Then splines on $\mathbb{Z}/p^2\mathbb{Z}$ are generated by the minimum generating set $B(R_G)$.

  Using the above results and Theorem 5.4 and corollary 5.5[21], we have shown that there exists a minimum generating set for wheel graphs over $\mathbb{Z}/p^2\mathbb{Z}$ for an arbitrary n.

5.3 Splines on Wheel graphs over $\mathbb{Z}/p^k\mathbb{Z}$

Splines on Wheel graphs over $\mathbb{Z}/p^k\mathbb{Z}$[46] Extending the above results of cycles to wheel graph W_{n+1} whose edges are labeled by some powers of $a \in R$,we have

- **Theorem**[46]

  Let a be a zero divisor in $\mathbb{Z}/m\mathbb{Z}$. Suppose all of the edges of W_{n+1} are labeled with powers of a and the set of edge labels is $\{a^{k_1}, a^{k_2}, \ldots, a^{k_n}\}$. Without loss of generality assume that a^{k_1} is the minimal power in the set and that a^{k_1} is the label on the edges l_n, l_{n+1}, $l_{n+2}, \ldots, l_{2n}$. Then the following matrix contains all generating $n+1$ splines on W_{n+1}.

$$B(R_{w_{n+1}}) = \begin{bmatrix} 1 & l_1 & \ldots & l_i & \ldots & l_{n-1} & l_n & l_n \\ 1 & l_1 & \ldots & l_i & \ldots & l_{n-1} & l_n & 0 \\ 1 & l_1 & \ldots & l_i & \ldots & l_{n-1} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots \\ 1 & l_1 & \ldots & 0 & \ldots & \vdots & \vdots & 0 \\ 1 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0 \end{bmatrix}$$

Here l_i is equal to the edge label a^{k_j} where $1 \leq j \leq n-1$.

Proof We need to verify that every element in our generating set is a spline on $R_{W_{n+1}}$and that every possible spline on $R_{W_{n+1}}$ can be written in terms of the generating set.

The trivial spline is a spline by definition. Notice that every other element of the form (l_i, l_i, ..., l_i, 0, ..., 0). The difference between any pair of adjacent vertices is 0 around every edge, except around the edges l_i and l_{2n}The spline conditions are trivially satisfied for each pair of adjacent vertices that differ by 0. The difference over the other two edges is l_i . Notice that l_i divides itself and the n^{th}, $(n+1)^{th}$,..., $(2n)^{th}$ edges are labeled by a^{k_1} by our assumption. Also l_n divides all other edge labels.Thus the spline conditions are satisfied at every edge.

Theorem 5.4[55] shows that every element f $\in R_{W_{n+1}}$ can be written as a linear combination of the splines in $B(R_{W_{n+1}})$.

Corollary 2.11[55] shows that the set is minimum proving the claim.

As a corollary to the above result,we have

- **Corollary**[46]

  Let W_{n+1} be a wheel graph let k be any positive integer. Then the splines on $R_{W_{n+1}}$ over $\mathbb{Z}/p^k\mathbb{Z}$ is the minimum generating set $B(R_{W_{n+1}})$ in the above result.

Proof The only possible edge labels over $\mathbb{Z}/p^k\mathbb{Z}$ are $\{(p), (p^2), (p^3), \ldots, (p^{k-1})\}$ By rotating the edge-labeled graph,we can assume that the edge l_n is labeled with the least power.This rotation induces an isomorphism on

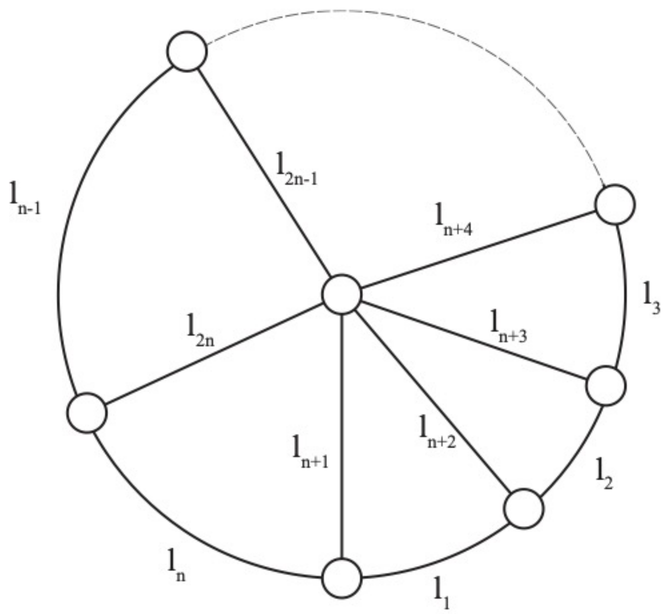

FIG. 5.3: Wheel Graph

the ring of splines. Thus the above result gives a minimum generating set for $R_{W_{n+1}}$ over $\mathbb{Z}/p^k\mathbb{Z}$.

- **Example**[46]
  We give a set of constant flow-up splines for wheel graph W_5.over $\mathbb{Z}/2^5\mathbb{Z}$.Then a minimum generating set for the splines over W_5 is $(B(R_{W_5})$ as follows

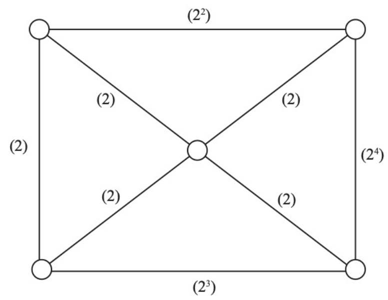

FIG. 5.4: Wheel Graph[46]

$$B(R_G) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2^3 \\ 2^3 \\ 2^3 \\ 2^3 \\ 0 \end{pmatrix} \begin{pmatrix} 2^4 \\ 2^4 \\ 2^4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2^2 \\ 2^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Now we give following theorem which gives a flow-up generating set for the generalized splines on C_n over the ring $\mathbb{Z}/m\mathbb{Z}$.

5.4 Splines on Cycle Graph over $\mathbb{Z}/m\mathbb{Z}$

Splines on Cycle Graph over $\mathbb{Z}/m\mathbb{Z}$[46]

- **Theorem**[46]
  Let C_n be a cycle and R be ring $\mathbb{Z}/m\mathbb{Z}$ where $m = m_1m_2$ where m_1, m_2 are prime factors of m.Let the edges of C_n be labeled by prime factors of m such that each prime factor of m appears at least once in the edge labeling of C_n. Then the following set is a flow-up generating set for R_{C_n}.

$$B(R_{C_n}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_{n-1}l_n \\ l_{n-2}l_{n-1} \\ \vdots \\ l_2l_3 \\ l_1l_2 \\ 0 \end{pmatrix} \begin{pmatrix} l_{n-1}l_n \\ \vdots \\ l_3l_4 \\ l_2l_3 \\ 0 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} l_{n-1}l_n \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

- **Remark**[46]
  When m has only two prime factors the generating set $B(R_{C_n})$ may not be minimum. It will lose a rank whenever two adjacent edges of C_n are labeled with distinct primes m_1 and m_2.We observe that when m has more than 2 prime factors, the generating set doesn't loose a rank since every vertex in cycles has only two edges attached to it.

  Proof First we note that each element in the set $B(R_{C_n})$ satisfies the edge conditions over the cycle C_n, and hence is a spline. Next, we want to show that every spline f in R_{C_n} can be expressed as a linear combination of elements in $B(R_{C_n})$.We use the method of induction for this, over the number of leading zeroes in f. If f has no leading zeros,then $f - f_{v_1}(1, 1, \ldots, 1)$ is a spline with one leading zero.Suppose f has i leading zeros. Then the restriction of f over the vertex v_{i+1}, i.e,$f_{v_{i+1}}$ is a multiple of l_il_{i+1}.Let $c_i = f_{v_{i+1}}/l_il_{i+1}$. Then, $f - c_i(0, 0, \ldots, 0, l_il_{i+1}, l_{i+1}l_{i+2}, \ldots, l_{n-1}l_n)$ is a spline with $i + 1$ leading zeroes. Thus the set $B(R_{C_n})$ forms a generating set for R_{C_n}.

- **Remark**[46]
  When m has only two prime factors the generating set B may not be minimum. It will lose a rank whenever two adjacent edges of C_n are labeled with distinct primes m_1 and m_2.We observe that when m has more than 2 prime factors, the generating set doesn't loose a rank since every vertex in cycles has only two edges.

However, if m has three or more prime factors and the edges of C_n are labeled such that each prime factor of m appears atleast once in edge labeling of C_n,then the above set $B(R_{C_n})$ will be minimum.

- **Example**[46]
  We give a set of flow-up splines for C_5 over $\mathbb{Z}/2.3.5\mathbb{Z}$ which forms a generating set for R_{C_5}.

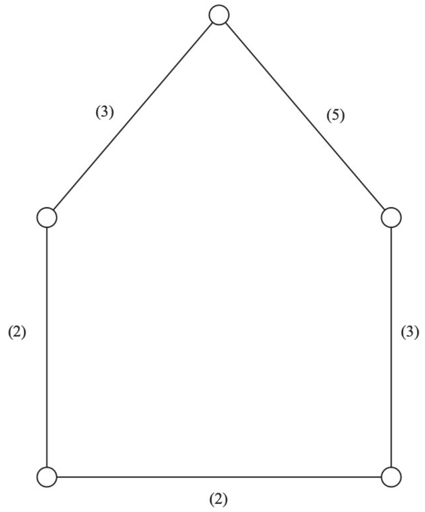

FIG. 5.5: Cycle Graph

$$B(R_{C_5}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 3.2 \\ 5.3 \\ 3.5 \\ 2.3 \\ 0 \end{pmatrix} \begin{pmatrix} 3.2 \\ 5.3 \\ 3.5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 3.2 \\ 5.3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 3.2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

- **Remark**[46]
  We observe that when m has more than two prime factors,the generating set doesn't loose a rank.

5.5 Splines on wheel graphs over $\mathbb{Z}/m\mathbb{Z}$

In this section we extend the method used in the previous section to construct a minimum generating set for the wheel graph W_{n+1} (Fig.5.3). We have the following theorem.

- **Theorem**[46]
  Let W_{n+1} be a wheel graph with $n+1$ vertices and consider the quotient ring $\mathbb{Z}/m\mathbb{Z}$

and $m = m_1m_2$ where m_1 and m_2 are primes.Label the graph in such a way that m_1 and m_2 appear atleast once as edge labeling.Then the following set $B(R_{W_{n+1}})$ is a generating set for $R_{W_{n+1}}$.

$$B(R_{W_{n+1}}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}\dots l_{2n} \\ l_{n-2}l_nl_{2n} \\ \vdots \\ l_2l_3l_{n+3} \\ l_1l_2l_{n+2} \\ 0 \end{pmatrix} \begin{pmatrix} l_{n+1}l_{n+2}\dots l_{2n} \\ l_{n-2}l_nl_{2n} \\ \vdots \\ l_2l_3l_{n+3} \\ 0 \\ 0 \end{pmatrix} \dots \begin{pmatrix} l_{n+1}l_{n+2}\dots l_{2n} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Proof Since the wheel graph is obtained by adding a vertex and the edges to the graph C_n, we want to show that the edge conditions are satisfied for the edges, l_{n+1},l_{n+2},... l_{2n}, by each spline in $B(R_{W_{n+1}}$. Since the last vertex v_{n+1} is labeled with the element $l_{n+1}l_{n+2}\dots l_{2n}$, which is a product of the labeling on the edges that are added to C_n to get the wheel graph.The difference of the vertex label v_{n+1} with any vertex v_i will be a multiple of l_{n+i}. This immediately proves our claim that the edge conditions are satisfied. Also, the above set generates any arbitrary spline f on $R_{W_{n+1}}$ by induction over the number of leading zeroes in the elements of $B(R_{W_{n+1}})$.

We observe that when $m = m_1m_2$ where m_1 and m_2 are primes,all splines on wheel graphs are trivial splines, whenever m_1 or m_2 both appear atleast once as edge labels for the edges adjacent at each of it's vertices.

- **Example**[46]

  We give an example of wheel graph with 6 vertices.

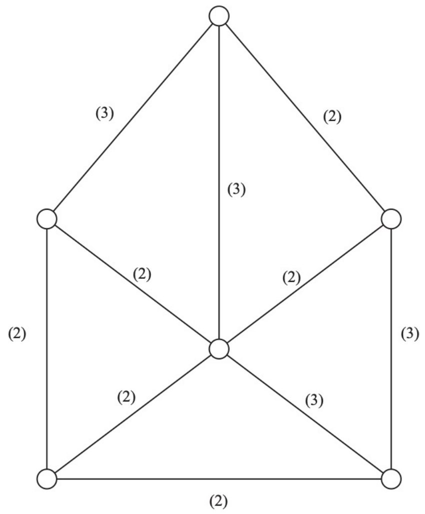

FIG. 5.6: Wheel Graph

The following set is generating set $B_{R_{W_6}}$ for W_6.

$$B_{R_{W_6}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2.3.2.3.2 \\ 3.2.2 \\ 3.3.2 \\ 2.2.3 \\ 2.3.3 \\ 0 \end{pmatrix} \begin{pmatrix} l \\ 3.2.2 \\ 3.3.2 \\ 2.2.3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2.3.2.3.2 \\ 3.2.2 \\ 3.3.2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2.3.2.3.2 \\ 3.2.2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2.3.2.3.2l \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

In this example, if at some vertex of the above graph, if all edges meeting at a point are either labeled as only m_1 or only m_2 the generating set will have non-trivial elements and hence the splines over the wheel graph will also be non-trivial.

Using similar algorithm, we can construct a generating set over a wheel graph, when m has more number of prime factors. Here we completely characterise the situation,when the generating set of the ring $R_{W_{n+1}}$ will be minimum.

- **Theorem**[46]
  Let W_{n+1} be a wheel graph, with vertices $v_1, v_2, \ldots, v_{2n}$ and edge labels $l_1, l_2, \ldots, l_{2n}$ and $m = m_1 m_2 \ldots m_r$ where $m_1, m_2, \ldots, m_r$ are primes. Let each edge of the wheel graph be labeled by the prime factors of m. Then, we get the same generating set $B(R_{W_{n+1}})$ as in case when $m = m_1 m_2$.The generating set will be minimum whenever the number of prime factors of m is greater than n, i.e, the number of vertices in the cycle graph C_n contained in the wheel graph W_{n+1}. When $r > n$,

  Proof We know that each element of generating set $B(R_{W_{n+1}})$ is the product of the labels of edges meeting at a vertex in wheel graph.[Refer the theorem in the case when $m = m_1 m_2$]

  Then the following set $B(R_{W_{n+1}})$ is a generating set for $R_{W_{n+1}}$, when $m = m_1 m_2 \ldots m_r$.

$$B(R_{W_{n+1}}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} l_{n+1} l_{n+2} \ldots l_{2n} \\ l_{n-2} l_n l_{2n} \\ \vdots \\ l_2 l_3 l_{n+3} \\ l_1 l_2 l_{n+2} \\ 0 \end{pmatrix} \begin{pmatrix} l_{n+1} l_{n+2} \ldots l_{2n} \\ l_{n-2} l_n l_{2n} \\ \vdots \\ l_2 l_3 l_{n+3} \\ 0 \\ 0 \end{pmatrix} \ldots \begin{pmatrix} l_{n+1} l_{n+2} \ldots l_{2n} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

The proof for minimum generating set follows from the fact that exactly three edges meet at a vertex lying in the cycle graph contained in the wheel graph and the interior vertex v_{n+1} is adjacent to exactly n edges in the wheel graph W_{n+1}.

- **Remark**[46]

  In wheel graphs when we exclude the central vertex, there are n vertices and we observe that if the number of prime factors in m are greater than or equal to n,i.e,$r \leq n$, then the generating set on wheel graphs looses rank over $\mathbb{Z}/m\mathbb{Z}$.

  Here we give an example of wheel graph when $r < n$ over $\mathbb{Z}/m\mathbb{Z}$.

- **Example**[46]

  Here we give an example of Wheel graph with 5 vertices when $m = 2.3.5$.i.e,the number of prime factors in m are less than the number of vertices in cycle graph contained in wheel graph.

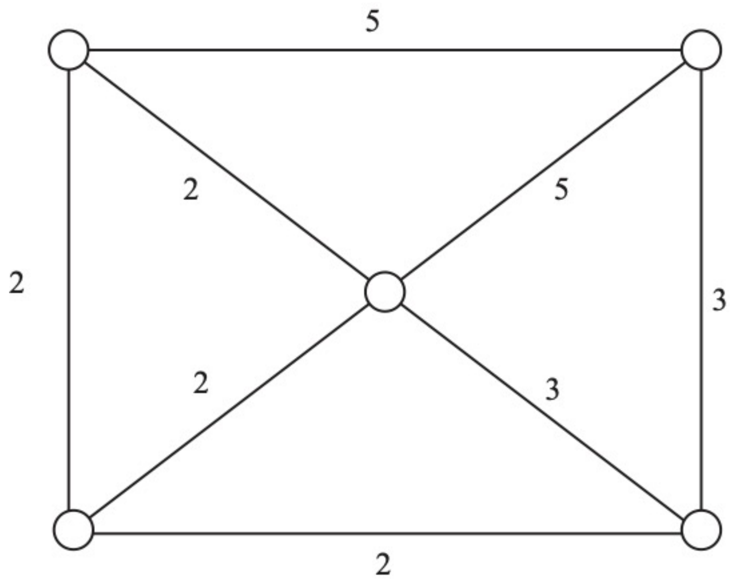

FIG. 5.7: Wheel Graph with 5 vertices[46]

Then the following set is generating set for R_{W_5}

$$B(R_{W_5}) = \left\{ \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \begin{pmatrix}2.3.5.2\\5.2.2\\3.5.5\\2.3.3\\0\end{pmatrix} \begin{pmatrix}2.3.5.2\\5.2.2\\3.5.5\\0\\0\end{pmatrix} \begin{pmatrix}2.3.5.2\\3.5.5\\0\\0\\0\end{pmatrix} \begin{pmatrix}2.3.5.2\\0\\0\\0\\0\end{pmatrix} \right\}$$

Generating set looses a rank when $m = 2.3.5$ on W_5 over $\mathbb{Z}/2.3.5\mathbb{Z}$.

Here we give an example of wheel graph when $r = n$ over $\mathbb{Z}/m\mathbb{Z}$.

- **Example**[46]
  Wheel graph with 5 vertices when $m = 2.3.5.7$.

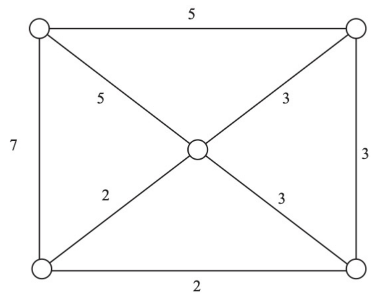

FIG. 5.8: Wheel Graph with 5 vertices[46]

Then the following set is generating set for R_{W_5}.

$$B(R_{W_5}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2.3.5.7 \\ 5.7.2 \\ 3.5.5 \\ 2.3.5 \\ 0 \end{pmatrix} \begin{pmatrix} 2.3.5.7 \\ 5.7.2 \\ 3.5.5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2.3.5.7 \\ 5.7.2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2.3.5.7 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Generating set looses a rank when $m = 2.3.5.7$.i.e,when number of prime factors of $m = n$,the number of vertices in cycle graph contained in wheel graph.

- **Remark**[46]
  In wheel graphs when we exclude the central vertex, there are n vertices and we observe that if the number of prime factors in m are greater than n, the splines on wheel graphs do not loose any rank over $\mathbb{Z}/m\mathbb{Z}$.

  Here we give an example of wheel graph when $r > n$ over $\mathbb{Z}/m\mathbb{Z}$.

- **Example**[46]
  Here we give wheel graph with 5 vertices when $m = 2.3.5.7.11$ The following set is

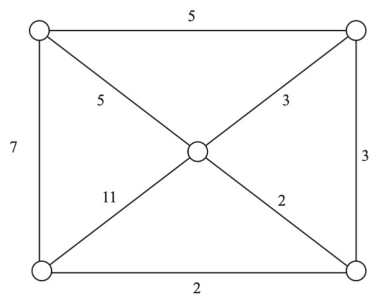

FIG. 5.9: Wheel Graph with 5 vertices

generating set for R_{W_5}

$$B(R_{W_5}) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 11.2.3.5 \\ 5.5.7 \\ 3.3.5 \\ 2.2.3 \\ 0 \end{pmatrix} \begin{pmatrix} 11.2.3.5 \\ 5.5.7 \\ 3.3.5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 11.2.3.5 \\ 5.5.7 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 11.2.3.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Clearly generating set doesn't loose any rank when $m = 2.3.5.7.11$.i.e,when the number of prime factors of m is greater then the number of vertices in cycle graph contained in the wheel graph.

5.6 Conclusions

Conclusions[46] We conclude our work with finding an algorithm for writing the generating set which acts as a basis for the generalized spline modules for cycle graphs, taking the base ring as the quotient ring of integers modulo m whenever $m = m_1m_2m_3\ldots m_r$, where each $m_i, i = 1, \ldots, r$, is a prime.The method is extendable to a generating set for the wheel graphs which is viewed as a graph extension to the cycle graph. Also, we noted that when the number of prime factors of m exceeds the number of vertices in the underlying cycle graph, the generating set is minimum. However, it may loose rank whenever m has fewer prime factors depending upon the labeling of the edges. This method is very systematic over the existing methods used in [21] and hence leads to number of open questions as to whether it can be extended to other family of graphs as well as base rings modulo powers of primes.