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The Optimality Conditions for Fuzzy Optimization Problem under the Concept of Generalized Convexity

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Abstract

In this paper we derive sufficient optimality conditions and obtain nondominated solutions for nonlinear constrained fuzzy optimization problem under the concept of generalized convexity and hukuhara diffrentiability of fuzzy-valued functions. To derive these conditions we use the partial order relation defined on fuzzy number spaces.

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1. Introduction

The optimization under a fuzzy environment or which involve fuzziness is called fuzzy optimization. In real process opimization, there exist different types of uncertainties in the system. For enhancing the capability of the optimization in an uncertain environment, the concept of fuzzy sets has been applied extensively.

Bellman and Zadeh (1970) introduced fuzzy optimization problems in [2] where they stated that a fuzzy decision can be viewed as the intersection of fuzzy goals and problem constraints. After that, various approaches to fuzzy linear and nonlinear optimization have been developed over the years by researchers.

A. A. K. Majumdar has proved the sufficient optimality conditions for multiobjective optimization problems using the concept of convexity and generalized convexity in his paper [13]. Wu has proved the sufficient optimality conditions for optimization problem with fuzzy-valued objective function and real constraints in [12] using pseudoconvexity of objective function. Using the approach of [13], in this paper we prove the sufficient optimality conditions for fuzzy optimization problem with fuzzy-valued objective function and fuzzy constraints, under the concept of convexity and generalized convexity of fuzzy-valued function includes more weaker convexity- quasiconvexity also and provide nondominated solution using partial order relation on fuzzy number spaces and hukuhara differentiability of fuzzy-valued function.

In Section 2, we introduce definitions, basic properties and arithmetics of fuzzy numbers. In Section 3, we consider the differential calculus of fuzzy-valued functions defined on $\mathbb{R}$ and $\mathbb{R}^n$ using the Hukuhara differentiability of fuzzy-valued functions. In Section 4, we define convexity and generalized convexity of fuzzy-valued function. In Section 5, we provide nondominated solutions of nonlinear fuzzy optimization problem by proving the sufficent optimality conditions and illustrate the results by two examples.

2. Fuzzy Numbers

Definition 2.1. [5] Let $\mathbb{R}$ be the set of real numbers and $\tilde{a} : \mathbb{R} \to [0, 1]$ be a fuzzy set on $\mathbb{R}$. We say that $\tilde{a}$ is a **fuzzy number** if it satisfies the following properties:

(i) $\tilde{a}$ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $\tilde{a}(x_0) = 1$;

(ii) $\tilde{a}$ is convex, that is, $\tilde{a}(tx + (1 - t)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$, whenever $x,y \in \mathbb{R}$ and $t \in [0, 1]$;

(iii) $\tilde{a}(x)$ is upper semicontinuous on $\mathbb{R}$, that is, $\{x/\tilde{a}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0, 1]$;

(iv) $\tilde{a}_0 = cl\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$ forms a compact set.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. For all $\alpha \in (0, 1]$, α-level set $\tilde{a}_\alpha$ of any $\tilde{a} \in F(\mathbb{R})$ is defined as $\tilde{a}_\alpha = \{x \in \mathbb{R}/\tilde{a}(x) \geq \alpha\}$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$. By the definition of fuzzy numbers, we can prove that, for any $\tilde{a} \in F(\mathbb{R})$ and for each $\alpha \in (0, 1]$, $\tilde{a}_\alpha$ is a compact convex subset of $\mathbb{R}$, and we write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$.

$\tilde{a} \in F(\mathbb{R})$ can be recovered from its α-cuts by a well-known decomposition theorem given in [6], which states that

$$\tilde{a} = \cup_{\alpha \in [0,1]} \alpha \cdot \tilde{a}_\alpha$$

where union on the right-hand side is the standard fuzzy union.

Definition 2.2. [17] According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space $F(\mathbb{R})$ by their α-cuts are as follows:

$$\begin{aligned}(\tilde{a} \oplus \tilde{b})_\alpha &= [\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U]\\ (\lambda \odot \tilde{a})_\alpha &= [\lambda \cdot \tilde{a}_\alpha^L, \lambda \cdot \tilde{a}_\alpha^U],\end{aligned}$$

where $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $\alpha \in [0, 1]$.

Definition 2.3. [10] We denote by K the set of all non-empty compact subset of $\mathbb{R}^n$. The Hausdorff metric d_H on K as defined in [8], is given by

$$d_H(A, B) = \max\{\sup_{x \in A} \inf_{y \in B} ||x - y||, \sup_{y \in B} \inf_{x \in A} ||x - y||\}.$$

For $A, B \in K$. Then the **metric** d_F on $F(\mathbb{R})$ is defined as

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0 \le \alpha \le 1} \{d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)\}$$

for all $\tilde{a}, \tilde{b} \in F(\mathbb{R})$.

Since $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are closed bounded intervals in $\mathbb{R}$,

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0 \le \alpha \le 1} \max\{|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L|, |\tilde{a}_\alpha^U - \tilde{b}_\alpha^U|\}.$$

In [4], it has been shown that $F(\mathbb{R})$ with the metric d_F is a metric space. Now here we define partial order relation on fuzzy number space $F(\mathbb{R})$.

Definition 2.4. For $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers in $F(\mathbb{R})$ and $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ and $\tilde{b}_\alpha = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U]$ are two closed intervals in $\mathbb{R}$, for all $\alpha \in [0, 1]$, we define

(i) $\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{a}_\alpha^L \le \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U \le \tilde{b}_\alpha^U$ for all $\alpha \in [0, 1]$;

(ii) $\tilde{a} \prec \tilde{b}$ if and only if

$$\begin{cases} \tilde{a}_\alpha^L < \tilde{b}_\alpha^L \\ \tilde{a}_\alpha^U \le \tilde{b}_\alpha^U \end{cases} \text{or} \begin{cases} \tilde{a}_\alpha^L \le \tilde{b}_\alpha^L \\ \tilde{a}_\alpha^U < \tilde{b}_\alpha^U \end{cases} \text{or} \begin{cases} \tilde{a}_\alpha^L < \tilde{b}_\alpha^L \\ \tilde{a}_\alpha^U < \tilde{b}_\alpha^U \end{cases}$$

for all $\alpha \in [0, 1]$.

Definition 2.5. [16] The membership function of a **triangular fuzzy number** $\tilde{a}$ is defined by

$$\zeta_{\tilde{a}}(r) = \begin{cases} \dfrac{(r - a^L)}{(a - a^L)} & if \;\; a^L \le r \le a \\ \dfrac{(a^U - r)}{(a^U - a)} & if \;\; a < r \le a^U \\ 0 & \text{otherwise} \end{cases}$$

which is denoted by $\tilde{a} = (a^L, a, a^U)$. The α-level set of $\tilde{a}$ is then

$$\tilde{a}_\alpha = [(1 - \alpha)a^L + \alpha a, (1 - \alpha)a^U + \alpha a].$$

3. Differential Calculus of Fuzzy-Valued Function

First we define continuity of fuzzy-valued function.

Definition 3.1. [9] Let V be a real vector space and $F(\mathbb{R})$ be a fuzzy number space. Then a function $\tilde{f} : V \to F(\mathbb{R})$ is called **fuzzy-valued function** defined on V.

Corresponding to such a function $\tilde{f}$ and $\alpha \in [0, 1]$, we define two real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ on V as $\tilde{f}_\alpha^L(x) = (\tilde{f}(x))_\alpha^L$ and $\tilde{f}_\alpha^U(x) = (\tilde{f}(x))_\alpha^U$ for all $x \in V$.

Definition 3.2. [4] Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. We say that $\tilde{f}$ is **continuous** at $c \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in \mathbb{R}$, with $\|x - c\| < \delta$, we have $d_F(\tilde{f}(x), \tilde{f}(c)) < \epsilon$.

Proposition 3.3. Let $\tilde{f} : \mathbb{R} \to F(\mathbb{R})$ be a fuzzy-valued function on $\mathbb{R}$. If $\tilde{f}$ is continuous at $c \in \mathbb{R}$, then functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are continuous at c for all $\alpha \in [0, 1]$.

Proof. The result follows using the definitions of continuity of fuzzy-valued function $\tilde{f}$ and metric on fuzzy numbers. ∎

Definition 3.4. [5] Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If there exists a fuzzy number $\tilde{c}$ such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$. Then $\tilde{c}$ is called **Hukuhara difference** of $\tilde{a}$ and $\tilde{b}$ and is denoted by $\tilde{a} \ominus_H \tilde{b}$.

Definition 3.5. [5] Let X be an open subset of $\mathbb{R}$. A fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is said to be **H-differentiable** at x_0 if there exists a fuzzy number $D\tilde{f}(x_0)$ such that the limits (with respect to metric d_F)

$$\lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x_0 + h) \ominus_H \tilde{f}(x_0)], \text{ and } \lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x_0) \ominus_H \tilde{f}(x_0 - h)]$$

both exist and are equal to $D\tilde{f}(x_0)$. In this case, $D\tilde{f}(x_0)$ is called the **H-derivative** of $\tilde{f}$ at x_0.

Remark 3.6. Many fuzzy-valued functions are H-differentaible for which Hukuhara differences $\tilde{f}(x^0 + h) \ominus_H \tilde{f}(x^0)$ and $\tilde{f}(x^0) \ominus_H \tilde{f}(x^0 - h)$ both exist. The following example illustrates the fact.

Example 3.7. Given in [15], let $\tilde{f} : (0, 2\pi) \to F(\mathbb{R})$ be defined on level sets by

$$[\tilde{f}(x)]_\alpha = (1 - \alpha)(2 + \sin(x))[-1, 1],$$

for $\alpha \in [0, 1]$. At $x^0 = \pi/2$, H-difference does not exist. Therefore, function is not H-differentiable at $x^0 = \pi/2$.

Proposition 3.8. Let X be an open subset of $\mathbb{R}$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at x_0 with derivative $D\tilde{f}(x_0)$, then $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are differentiable at x_0 for all $\alpha \in [0, 1]$. Moreover, we have $(D\tilde{f}(x_0))_\alpha = [D(\tilde{f}_\alpha^L)(x_0), D(\tilde{f}_\alpha^U)(x_0)]$.

Proof. The result follows from definitions of H-differentiability of fuzzy-valued function and metric on fuzzy number space. ■

Definition 3.9. [11] Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset X of $\mathbb{R}^n$ and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ be fixed.

(i) We say that $\tilde{f}$ has the i^{th} **partial H-derivative** $D_i \tilde{f}(\bar{x})$ at $\bar{x}$ if the fuzzy-valued function $\tilde{g}(x_i) = \tilde{f}(\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n)$ is H-differentiable at $\bar{x}_i$ with H-derivative $D_i \tilde{f}(\bar{x})$. We also write $D_i \tilde{f}(\bar{x})$ as $(\partial \tilde{f}/\partial x_i)(\bar{x})$.

(ii) We say that $\tilde{f}$ is **H-differentiable** at $\bar{x}$ if one of the partial H-derivatives $\partial \tilde{f}/\partial x_1, \ldots, \partial \tilde{f}/\partial x_n$ exists at $\bar{x}$ and the remaining n-1 partial H-derivatives exist on some neighborhoods of $\bar{x}$ and are continuous at $\bar{x}$ (in the sense of fuzzy-valued function).

Proposition 3.10. Let X be an open subset of $\mathbb{R}^n$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at $\bar{x} \in X$. Then $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also differentiable at $\bar{x} \in X$, for all $\alpha \in [0, 1]$. Moreover, $(D_i \tilde{f}(\bar{x}))_\alpha = [D_i \tilde{f}^L_\alpha(\bar{x}), D_i \tilde{f}^U_\alpha(\bar{x})]$, $i = 1, \ldots, n$.

Proof. The result follows from Proposition 3.3 and 3.8. ■

4. Generalized Convexity of Fuzzy-Valued Function

First we define generalized convexity for real-valued function.

Definition 4.1. [1, 13]

(i) Let $f : T \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex at $x = x_0 \in T$ if and only if T is convex and

$$f(\lambda x_0 + (1-\lambda)x) \le \lambda f(x_0) + (1-\lambda) f(x),$$

for all $x \in T$ and $\lambda \in [0, 1]$.

$f(x)$ is called strictly convex at $x = x_0$ if and only if the strict inquality sign holds in the above inequality. Also, if $f(x)$ is differentiable at $x_0 \in T$, then $f(x)$ is convex at $x = x_0$ if and only if

$$(\nabla f(x_0))^t (x - x_0) \le f(x) - f(x_0),$$

for all $x \in T$.

(ii) $f(x)$ is said to be quasiconvex at $x_0 \in T$, with T an arbitrary set, if and only if

$$f(x) \le f(x_0) \implies f(\lambda x + (1-\lambda)x_0) \le f(x_0),$$

for all $x \in T$ and $\lambda \in [0, 1]$.

$f(x)$ is said to be quasiconvex on T if and only if it is quasiconvex at each $x \in T$. Furthermore, if $f(x)$ is differentiable at $x_0 \in T$, $f(x)$ is quasiconvex at $x = x_0$ if and only if

$$f(x) \le f(x_0) \implies (\nabla f(x_0))^t (x - x_0) \le 0, \ for\ all\ x \in T.$$

(iii) A function $f(x)$ defined on an open set $T \subseteq \mathbb{R}^n$, is said to be pseudoconvex at $x_0 \in T$ (on T), if and only if it is differentiable at x_0 (at each point of T) with

$$\nabla f(x_0)^t (x - x_0) \ge 0 \implies f(x) \ge f(x_0);$$

or equivalently

$$f(x) < f(x_0) \implies \nabla(f(x_0))^t (x - x_0) < 0,$$

for all $x \in T$ (for all $x_0 \in T$).

Definition 4.2. [1, 13] A function $f(x)$, defined on an open set $T (\subseteq \mathbb{R}^n)$, is said to be strictly pseudoconvex at $x_0 \in T$ (on T) if and only if it is differentiable at x_0 (at each point in T) with

$$(\nabla f(x_0))^t (x - x_0) \ge 0 \implies f(x) > f(x_0);$$

or equivalently

$$f(x) \le f(x_0) \implies \nabla(f(x_0))^t (x - x_0) < 0,$$

for each x $(\neq x_0) \in T$ (for each $x_0 \in T$).

Now we define here generalized convexity of fuzzy-valued function.

Definition 4.3. Let $\tilde{f} : T \subseteq \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function and T be a convex set. We say that $\tilde{f}$ is convex at $x_0 \in T$ if

$$\tilde{f}(\lambda x_0 + (1 - \lambda)x) \preceq (\lambda \odot \tilde{f}(x_0) \oplus ((1 - \lambda) \odot \tilde{f}(x))$$

for each $\lambda \in (0, 1)$ and each $x \in T$.

Proposition 4.4. [11] Let $\tilde{f} : T \subseteq \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function and T be a convex set. Then $\tilde{f}$ is convex at $x_0 \in T$ if and only if $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are convex at x_0 for all $\alpha \in [0, 1]$.

Definition 4.5. $\tilde{f} : T \to F(\mathbb{R})$ be a fuzzy-valued function defined on a arbitrary set $T \subseteq \mathbb{R}^n$. We say that $\tilde{f}$ is qausiconvex at x_0 if and only if the real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are quasiconvex at x_0 for all $\alpha \in [0, 1]$.

Definition 4.6. $\tilde{f} : T \to F(\mathbb{R})$ be a fuzzy-valued function defined on an open set $T \subseteq \mathbb{R}^n$. We say that $\tilde{f}$ is pseudoconvex (strictly pseudoconvex) at x_0 if and only if the real-valued functions $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are pseudoconvex (strictly pseudoconvex) at x_0 for all $\alpha \in [0, 1]$.

Example 4.7. Consider the fuzzy-valued function $\tilde{f} : \mathbb{R} \to F(\mathbb{R})$ defined by

$$\tilde{f}(x) = (\tilde{1} \odot x) \oplus (\tilde{2} \odot x^2),$$

where $\tilde{1} = (0, 1, 2)$ and $\tilde{2} = (1, 2, 3)$ are triangular fuzzy numbers. Then $\tilde{f}$ is strictly convex fuzzy-valued function, since $\tilde{f}_\alpha^L(x) = \alpha x^2 + (1+\alpha)x$ and $\tilde{f}_\alpha^U(x) = (2-\alpha)x^2 + (3-\alpha)x$ are strictly convex functions for $\alpha \in [0, 1]$.

We consider here an example of quasiconvex fuzzy-valued function as follows.

Example 4.8. Consider the fuzzy-valued function $\tilde{f} : \mathbb{R} \to F(\mathbb{R})$ defined by

$$\tilde{f}(x) = \sqrt{|\tilde{1} \odot x|},$$

where $\tilde{1} = (0, 1, 2)$ is triangular fuzzy number. Squareroot and absolute value of fuzzy number is defined in [14]. Then $\tilde{f}$ is quasiconvex fuzzy-valued function on $\mathbb{R}$, since $\tilde{f}_\alpha^L(x) = \sqrt{|\alpha x|}$ and $\tilde{f}_\alpha^U(x) = \sqrt{|(2-\alpha)x|}$ are quasiconvex functions for $\alpha \in [0, 1]$.

5. Optimality Conditions for Fuzzy Optimization Problem

Consider the following nonlinear fuzzy optimization problem

$$(FOP) \quad Minimize\ \ \tilde{f}(x) = \tilde{f}(x_1, \ldots, x_n)$$

$$Subject\ to\ \ \tilde{g}_j(x) \preceq \tilde{0}, \quad j = 1, \ldots, m,$$

$$x \in T \subseteq \mathbb{R}^n.$$

where T is open set and $\tilde{f}$ and $\tilde{g}_j$, $j = 1, \ldots, m$, are fuzzy-valued functions defined on T. For (FOP), the solution is defined in terms of a (weak) nondominated solution in the following sense.

Definition 5.1. Let $x_0 \in X = \{x \in T : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, \ldots, m\}$. We say that an x_0 is a nondominated solution of (FOP) if and only if there exists no $x_1 (\neq x_0) \in X$ such that $\tilde{f}(x_1) \preceq \tilde{f}(x_0)$. It is said to be a weak nondominated solution if and only if there exists no $x_1 \in X$ such that $\tilde{f}(x_1) \prec \tilde{f}(x_0)$.

Using the concept of convexity and generalized convexity of fuzzy-valued function defined in previous section, we prove here sufficient optimality conditions and obtain the nondominated solution for (FOP).

Theorem 5.2. Assume that an $x_0 \in X$ satisfies the following conditions (i)–(iii):

(i) $\tilde{f}(x)$, $\tilde{g}_j(x)$, j= 1,…,m, are H-differentiable at $x = x_0 \in X$;

(ii) $\tilde{f}(x)$, $\tilde{g}_j(x)$, j= 1,…,m, are convex at $x = x_0 \in X$;

(iii) there exist $0 \le \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that

(a) $\nabla \tilde{f}^L_\alpha(x_0) + \nabla \tilde{f}^U_\alpha(x_0) + \sum_{j=1}^{m} \nabla \tilde{g}^U_{j0}(x_0) \cdot \mu_j = 0$, for all $\alpha \in [0, 1]$;

(b) $\mu_j \cdot \tilde{g}^U_{j0}(x_0) = 0$, for all $j = 1, \ldots, m$.

Then, x_0 is a weak nondominated sotuion of (FOP).

Proof. Suppose that $x_0 \in X$ is not weak nondominated. Then there exists $x_1 \in X$ such that $f(x_1) \prec f(x_0)$. That is, there exists $x_1 \in X$ such that

$$\begin{cases} \tilde{f}^L_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) \\ \tilde{f}^U_\alpha(x_1) \le \tilde{f}^U_\alpha(x_0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(x_1) \le \tilde{f}^L_\alpha(x_0) \\ \tilde{f}^U_\alpha(x_1) < \tilde{f}^U_\alpha(x_0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) \\ \tilde{f}^U_\alpha(x_1) < \tilde{f}^U_\alpha(x_0) \end{cases}$$

for all $\alpha \in [0, 1]$. Therefore

$$\tilde{f}^L_\alpha(x_1) + \tilde{f}^U_\alpha(x_1) < \tilde{f}^L_\alpha(x_0) + \tilde{f}^U_\alpha(x_0),$$

for all $\alpha \in [0, 1]$. That is,

$$F_\alpha(x_1) - F_\alpha(x_0) < 0,$$

where $F_\alpha(x) = \tilde{f}^L_\alpha(x) + \tilde{f}^U_\alpha(x)$, for all $\alpha \in [0, 1]$. By definition of partial ordering, we have

$$\begin{aligned} X &= \{x \in T \subset \mathbb{R}^n : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, \ldots, m\} \\ &= \{x \in T \subset \mathbb{R}^n : \tilde{g}^L_{j\alpha}(x) \le 0 \; and \; \tilde{g}^U_{j\alpha}(x) \le 0, j = 1, \ldots, m\} \\ &= \{x \in T \subset \mathbb{R}^n : \tilde{g}^U_{j\alpha}(x) \le 0, j = 1, \ldots, m\} \\ &= \{x \in T \subset \mathbb{R}^n : \tilde{g}^U_{j0}(x) \le 0, j = 1, \ldots, m\} \end{aligned}$$

Since $x_0, x_1 \in X$, we have

$$\tilde{g}^U_{j0}(x_1) - \tilde{g}^U_{j0}(x_0) \le 0$$

where $j \in J = \{j : \tilde{g}^U_{j0}(x_0) = 0\}$ is an index set of acitve constraints at $x = x_0$. By assumption **(ii)** of the Theorem, $\nabla F_\alpha(x_0)(x_1 - x_0) < 0$, $\nabla \tilde{g}^U_{j0}(x_0)(x_1 - x_0) \le 0$, $j \in J$ and for all $\alpha \in [0, 1]$.

Thus, the following system of inequalities $\nabla F_\alpha(x_0)z < 0$ for all $\alpha \in [0, 1]$, $\nabla \tilde{g}^U_{j0}(x_0)z \le 0$ possess a solution $z = x_1 - x_0$.

Therefore, by Tucker's theorem of alternatives (ref. [1]), there exist no $\lambda > 0$ and $\mu'_j \ge 0$ such that

$$\nabla F_\alpha(x_0)\lambda + \sum_{j \in J} \nabla \tilde{g}^U_{j0}(x_0) \cdot \mu'_j = 0,$$

for all $\alpha \in [0, 1]$. That is,

$$\nabla \tilde{f}_\alpha^L(x_0) + \nabla \tilde{f}_\alpha^U(x_0) + \sum_{j \in J} \nabla \tilde{g}_{j0}^U(x_0) \cdot \mu_j = 0,$$

where $\mu_j = \mu_j^{'}/\lambda$ and $F_\alpha(x_0) = \tilde{f}_\alpha^L(x_0) + \tilde{f}_\alpha^U(x_0)$, for all $\alpha \in [0, 1]$.

Since J is an index set of active constraints, we have $\tilde{g}_{j0}^U(x_0) \neq 0$ for $j \notin J$. Also $\tilde{g}_{j0}^U(x_0) \cdot \mu_j = 0$ for $j = 1, \ldots, m$ by **(iii)(b)** of the Thereom

$$\mu_j = 0 \; for \; j \notin J.$$

Therefore

$$\sum_{j \in J} \nabla \tilde{g}_{j0}^U(x_0) \cdot \mu_j = \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(x_0) \cdot \mu_j.$$

Therefore, there exist no $\mu_j \geq 0$, $j = 1, \ldots, m$ such that

$$\nabla \tilde{f}_\alpha^L(x_0) + \nabla \tilde{f}_\alpha^U(x_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(x_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0, 1]$.

Which contradict to **(iii)(a)** of the Theorem. Hence, x_0 is a weak nondominated solution of (FOP). ■

Theorem 5.3. Assume that an $x_0 \in X$ satisfies the following conditions (i)–(iii):

(i) $\tilde{f}(x)$, $\tilde{g}_j(x)$, $j = 1, \ldots, m$, are strictly pseudoconvex at $x = x_0 \in X$;

(ii) there exist $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that

(a) $\nabla \tilde{f}_\alpha^L(x_0) + \nabla \tilde{f}_\alpha^U(x_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^U(x_0) \cdot \mu_j = 0$, for all $\alpha \in [0, 1]$;

(b) $\mu_j \cdot \tilde{g}_{j0}^U(x_0) = 0$, for all $j = 1, \ldots, m$.

Then, x_0 is a nondominated sotuion of (FOP).

Proof. Suppose x_0 is not nondominated solution, then there exists an $x_1 \in X$ such that $\tilde{f}(x_1) \preceq \tilde{f}(x_0)$. i.e., $\tilde{f}_\alpha^L(x_1) \leq \tilde{f}_\alpha^L(x_0)$ and $\tilde{f}_\alpha^U(x_1) \leq \tilde{f}_\alpha^U(x_0)$ for all $\alpha \in [0, 1]$.

By assumption of strict pseudoconvexity of the function $\tilde{f}(x)$ at $x = x_0$, we have $\tilde{f}_\alpha^L(x)$ and $\tilde{f}_\alpha^L(x)$ are also strict pseudoconvex functions. Using the above inqualities, we obtain $\nabla \tilde{f}_\alpha^L(x_0)^t(x_1 - x_0) < 0$ and $\nabla \tilde{f}_\alpha^U(x_0)^t(x_1 - x_0) < 0$, for all $\alpha \in [0, 1]$.

Furthermore, we have

$$\tilde{g}_{j0}^{U}(x_1) - \tilde{g}_{j0}^{U}(x_0) \leq 0$$

where $j \in J = \{j : \tilde{g}_{j0}^{U}(x_0) = 0\}$ is an index set of acitve constraints at $x = x_0$. Therefore, we have

$$\nabla F_{\alpha}(x_0)^t (x_1 - x_0) < 0$$

and

$$\nabla \tilde{g}_{j0}^{U}(x_0)^t (x_1 - x_0) \leq 0$$

where $F_{\alpha}(x_0) = \tilde{f}_{\alpha}^{L}(x_0) + \tilde{f}_{\alpha}^{U}(x_0)$, for all $\alpha \in [0, 1]$. Thus, the following system of inequalities $\nabla F_{\alpha}(x_0)^t z < 0$, $\nabla \tilde{g}_{j0}^{U}(x_0)^t z < 0$ possess a soltuion $z = x_1 - x_0$ for $\alpha \in [0, 1]$.

Therefore, by the Gordan theorem of alternatives (ref. [1]), there exist no $\lambda > 0$ and $0 \leq \mu_j^{'} \in \mathbb{R}$, $j = 1, \ldots, m$, such that

$$\nabla F_{\alpha}(x_0)\lambda + \sum_{j \in J} \nabla \tilde{g}_{j0}^{U}(x_0) \cdot \mu_j^{'} = 0,$$

for all $\alpha \in [0, 1]$. That is

$$\nabla \tilde{f}_{\alpha}^{L}(x_0) + \nabla \tilde{f}_{\alpha}^{U}(x_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^{U}(x_0) \cdot \mu_j = 0,$$

for all $\alpha \in [0, 1]$, violating assumption **(ii)(a)** of the theorem.

Hence, x_0 is a nondominated solution of (FOP). ■

Theorem 5.4. Assume that an $x_0 \in X$ satisfies the following conditions (i)–(iii):

(i) $\tilde{f}(x)$ is pseudoconvex at $x = x_0 \in X$;

(ii) $\tilde{g}_j(x)$ are quasiconvex and differentiable at x_0, for $j = 1, \ldots, m$;

(iii) there exist $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that

(a) $\nabla \tilde{f}_{\alpha}^{L}(x_0) + \nabla \tilde{f}_{\alpha}^{U}(x_0) + \sum_{j=1}^{m} \nabla \tilde{g}_{j0}^{U}(x_0) \cdot \mu_j = 0$, for all $\alpha \in [0, 1]$;

(b) $\mu_j \cdot \tilde{g}_{0j}^{U}(x_0) = 0$, $j = 1, \ldots, m$.

Then, x_0 is a nondominated sotuion of (FOP).

Proof. We can prove this result similarly as proof of above theorems. ■

Now we prove sufficient optimality condition for (FOP) under quasiconvexity of fuzzy-valued objective function.

Theorem 5.5. Assume that, for $x_0 \in X = \{x \in T : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, \ldots, m\}$.

(i) $\tilde{f}, \tilde{g}_j$, $j = 1, \ldots, m$ are H-differentiable at x_0.

(ii) for $j \in J$, where $J = \{j | \tilde{g}^U_{j0}(x) = 0, j = 1, \ldots, m\}$, $j \neq s$, $\tilde{g}^U_{j0}$ is quasiconvex; for $j = s$, $\tilde{g}^U_{s0}$ is strictly pseudoconvex and $\tilde{f}$ is quasiconvex.

Let $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$ and $x_0 \in X$ satisfies the following conditions:

(1) $\nabla \tilde{f}^L_\alpha(x_0) + \nabla \tilde{f}^U_\alpha(x_0) + \sum_{j=1}^{m} \nabla \tilde{g}^U_{j0}(x_0) \cdot \mu_j = 0$, for all $\alpha \in [0, 1]$;

(2) $\mu_j \cdot \tilde{g}^U_{0j}(x_0) = 0$, $j = 1, \ldots, m$.

Then, x_0 is a nondominated solution of (FOP).

Proof. Suppose x_0 is not nondominated solution, then there exists an $x_1 \in X$ such that $\tilde{f}(x_1) \preceq \tilde{f}(x_0)$. That is

$$\tilde{f}^L_\alpha(x_1) \leq \tilde{f}^L_\alpha(x_0) \text{ and} \tilde{f}^U_\alpha(x_1) \leq \tilde{f}^U_\alpha(x_0) \tag{5.1}$$

for all $\alpha \in [0, 1]$. By assumption (i) and (ii) of Theorem, $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are differentiable and quasiconvex functions for $\alpha \in [0, 1]$. Moreover,

$$\begin{aligned} X &= \{x \in T : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, \ldots, m\} \\ & \{x \in T : \tilde{g}^U_{j0}(x) \preceq \tilde{0}, j = 1, \ldots, m\} \end{aligned}$$

Therefore, using Theorem 2.1 of [7], we say that $x_0 \in X$ is a global minimum of $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ for $\alpha \in [0, 1]$. That is, $\tilde{f}^L_\alpha(x_0) \leq \tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x_0) \leq \tilde{f}^U_\alpha(x)$ for all $x \in X$ and $\alpha \in [0, 1]$. This contradicts to inequalities (5.1). This completes the proof. ∎

Here we provide two examples to show the effect of fuzzy modelling of the following crisp type optimization problem.

Example 5.6.

$$Minimize \quad f(x_1, x_2) = 2 \cdot x_1^2 + 2 \cdot x_2^2$$

$$Subject\ to\ :\ g(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \leq 3$$

has the minimum point $(x_1^*, x_2^*) = (2 - \sqrt{3}/\sqrt{2}, 2 - \sqrt{3}/\sqrt{2})$ and minimum value is $f(x_1^*, x_2^*) = 6.419$.

Now we consider a fuzzy optimization problem having fuzzy coefficients and we find the nondominated solution using the otpimality conditions.

Example 5.7. We consider the following fuzzy optimization problem

$$Minimize \quad \tilde{f}(x_1, x_2) = (\tilde{2} \odot x_1^2) \oplus (\tilde{2} \odot x_2^2)$$

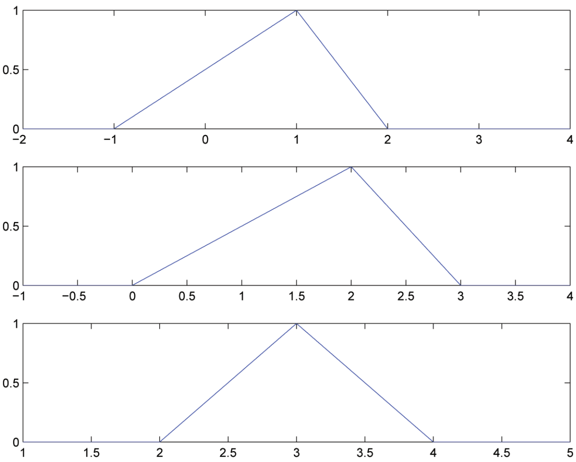

Figure 1: Membership functions of triangular fuzzy numbers $\tilde{1} = (-1, 1, 2)$, $\tilde{2} = (0, 2, 3)$ and $\tilde{3} = (2, 3, 4)$.

$$Subject\ to:\ \tilde{g}(x_1, x_2) = (\tilde{1} \odot (x_1 - 2)^2) \oplus (\tilde{1} \odot (x_2 - 2)^2) \preceq \tilde{3}$$

where $\tilde{2} = (0, 2, 3)$, $\tilde{1} = (-1, 1, 2)$ and $\tilde{3} = (2, 3, 4)$ are triangular fuzzy numbers as shown in figure 1.

By arithmetics of fuzzy numbers, we obtain $\tilde{f}^L_\alpha(x_1, x_2) = 2\alpha x_1^2 + 2\alpha x_2^2$, $\tilde{f}^U_\alpha(x_1, x_2) = (3-\alpha)x_1^2 + (3-\alpha)x_2^2$ and $\tilde{g}^U_\alpha(x_1, x_2) = (2-\alpha)(x_1-2)^2 + (2-\alpha)(x_2-2)^2 \leq (4-\alpha)$.

We also obtain

$\nabla\tilde{f}^L_\alpha(x_1, x_2) = \begin{pmatrix} 4\alpha x_1 \\ 4\alpha x_2 \end{pmatrix}$,

$\nabla\tilde{f}^U_\alpha(x_1, x_2) = \begin{pmatrix} 2(3-\alpha)x_1 \\ 2(3-\alpha)x_2 \end{pmatrix}$ and

$\nabla\tilde{g}^U_0(x_1, x_2) = \begin{pmatrix} 2(x_1 - 2) \\ 2(x_2 - 2) \end{pmatrix}$.

For checking the conditions (a) and (b) in Theorem 5.3, we need to solve the following system of equations:

$$\alpha x_1 + 3x_1 + 2\mu x_1 - 4\mu = 0$$

$$\alpha x_2 + 3x_2 + 2\mu x_2 - 4\mu = 0$$

$$\mu \cdot ((x_1 - 2)^2 + (x_2 - 2)^2 - 2) = 0.$$

Then we get $(x_1, x_2) = (1, 1)$ and $\mu = (\alpha + 3)/2$. We see that $(x_1, x_2) = (1, 1)$ is the feasible solution to the given (FOP).

For any fixed $\alpha \in [0, 1]$, we see that $\tilde{f}^L_\alpha$, $\tilde{f}^U_\alpha$, $\tilde{g}^L_\alpha$ and $\tilde{g}^U_\alpha$ are strictly convex, i.e., striclty pseudoconvex. Therefore from Theorem 5.3, we say that $(x_1^*, x_2^*) = (1, 1)$ is nondominated solution to the given (FOP) and minimum value of the fuzzy-valued objective function is $\tilde{4} = (0, 4, 6)$ having $\tilde{4}_\alpha = [4\alpha, 6-2\alpha]$. We defuzzify the minimum

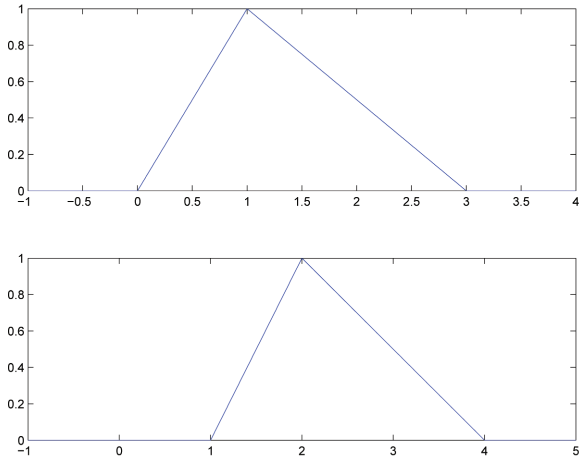

Figure 2: Membership functions of triangular fuzzy numbers $\tilde{1} = (0, 1, 3)$ and $\tilde{2} = (1, 2, 4)$.

value using the center of area method given in [6] as 3.3333. If we compare with this solution with a solution to crisp type of optimization problem in example4.8 which is 6.419. We observe that by approximating coefficients as fuzzy numbers we get better minimum value.

Remark 5.8. In the above example, we have solved fuzzy optimization problem having fuzzy coefficients are nonsymmetric left spread triangular fuzzy numbers. If we spread nonsymmetric triangular fuzzy numbers on right side as shown in the following figure 2, then nondominated solution of the same fuzzy optimization problem is given by $(x_1^*, x_2^*) = (2 - 2/\sqrt{6}, 2 - 2/\sqrt{6})$ and minimum value is $\tilde{f}(x_1^*, x_2^*) = (2.8, 5.6, 11.2)$. Its defuzzified value is 6.533.

If we consider fuzzy coefficients are symmetric triangular fuzzy numbers as shown in figure 3,then the nondominated solution will be $(x_1^*, x_2^*) = (1, 1)$ and $\mu = 4$. In this case, minimum value is $\tilde{4} = (2, 4, 6)$ and its defuzzified value is 4.

Thus fuzzification of the parameters representing coefficients of the x_1^2 and x_2^2 in f and coefficients of $(x_1 - 2)^2$, $(x_2 - 2)^2$ in g has a significant effect on the nondominated solution and the defuzzified value of the objective function.

Example 5.9. Let $\tilde{f}$ and $\tilde{g}$ are fuzzy-valued functions defined on $\mathbb{R}^2$ as

$$\tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^2) \oplus (\tilde{2} \odot x_2^2) \oplus ((\tilde{-3}) \odot x_1) \oplus ((\tilde{-3}) \odot x_2)$$

and

$$\tilde{g}(x_1, x_2) = (\tilde{3} \odot x_1) \oplus (\tilde{5} \odot x_2) \oplus (\tilde{-7})$$

where $\tilde{1} = (0, 1, 2)$, $(\tilde{-3}) = (-4, -3, -2)$, $\tilde{3} = (2, 3, 4)$, $\tilde{5} = (4, 5, 6)$ and $(\tilde{-7}) = (-8, -7, -6)$ are triangular fuzzy numbers.

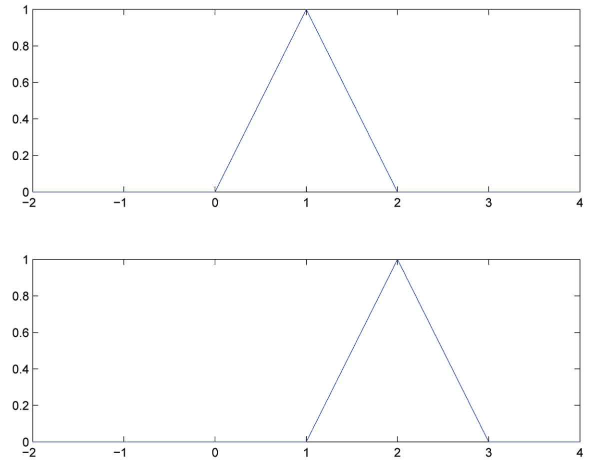

Figure 3: Membership functions of triangular fuzzy numbers $\tilde{1} = (0, 1, 2)$ and $\tilde{2} = (1, 2, 3)$.

Using the above fuzzy-valued functions, we consider the following fuzzy optimization problem

$$min \;\; \tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^2) \oplus (\tilde{2} \odot x_2^2) \oplus ((\tilde{-3}) \odot x_1) \oplus ((\tilde{-3}) \odot x_2)$$

$$subject\ to\ \ constraint : \; \tilde{g}(x_1, x_2) = (\tilde{3} \odot x_1) \oplus (\tilde{5} \odot x_2) \oplus (\tilde{-7}) \preceq \tilde{0}$$

where $\tilde{0} = (0, 0, 0)$.

Using arithmetics of fuzzy numbers, we obtain

$$\tilde{f}_\alpha^L(x_1, x_2) = \alpha x_1^2 + (1 + \alpha)x_2^2 + (-4 + \alpha)x_1 + (-4 + \alpha)x_2$$

and

$$\tilde{f}_\alpha^U(x_1, x_2) = (2 - \alpha)x_1^2 + (3 - \alpha)x_2^2 + (-2 - \alpha)x_1 + (-2 - \alpha)x_2.$$

and

$$\tilde{g}_\alpha^L(x_1, x_2) = (2 + \alpha)x_1 + (4 + \alpha)x_2 + (-8 + \alpha)$$

and

$$\tilde{g}_\alpha^U(x_1, x_2) = (4 - \alpha)x_1 + (6 - \alpha)x_2 + (-6 - \alpha).$$

Now we have

$$\nabla \tilde{f}_\alpha^L(x_1, x_2) = \begin{pmatrix} 2\alpha x_1 + (-4 + \alpha) \\ 2(1 + \alpha)x_2 + (-4 + \alpha) \end{pmatrix},$$

$$\nabla \tilde{f}_\alpha^U(x_1, x_2) = \begin{pmatrix} 2(2 - \alpha)x_1 - 2 - \alpha \\ 2(3 - \alpha)x_2 - 2 - \alpha \end{pmatrix},$$

$$\nabla \tilde{g}_0^U(x_1, x_2) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}.$$

For checking the conditions (a) and (b) in Theorem 5.4, we solve the following system of equations:

$$4x_1 - 6 + 4\mu = 0$$

$$8x_2 - 6 + 6\mu = 0$$

$$\mu \cdot (4x_1 + 6x_2 - 6) = 0.$$

Then, we get $(x_1, x_2) = \left(\frac{33}{34}, \frac{6}{17}\right)$ and $\mu = \frac{9}{17}$ which is feasible solution to the given (FOP).

For any fixed $\alpha \in [0, 1]$, we see that $\tilde{f}^L_\alpha$, $\tilde{f}^U_\alpha$ are strictly convex, i.e., pseudoconvex. Also, $\tilde{g}^L_\alpha$ and $\tilde{g}^U_\alpha$ are convex. i.e., quasiconvex. Therefore, by Theorem 5.4 we say that $(x_1, x_2) = \left(\frac{33}{34}, \frac{6}{17}\right)$ is nondominated solution.

6. Conclusion

In this paper, we have defined the concept of generalized convexity of fuzzy-valued functions. Using these, we have proved the sufficient conditions for x to be the nondominated solution of fuzzy optimization problem. We have also provided the two examples to illustrates the application of the theorems and have shown then the fuzzification of crisp otpimization problem does have significant effect on the solution.

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