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FIRST AND SECOND-ORDER OPTIMALITY CONDITIONS FOR UNCONSTRAINED L-FUZZY OPTIMIZATION PROBLEMS

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Abstract. Based on the concept of parametric total order relation on L-fuzzy numbers defined by S. Saito and H. Ishi [9], this paper establishes the first and second order necessary and sufficient optimality conditions for unconstrained multi-variable L-fuzzy optimization problems.

1. Introduction

The concept of fuzzy sets was introduced by L.A.Zadeh (1965). After this, many applications of fuzzy sets have been developed. One of them is fuzzy optimization, which accounts for any imprecision in the optimization problems. Bellman and Zadeh(1970) introduced fuzzy optimization problems in [1] where they have stated that a fuzzy decision can be viewed as the intersection of fuzzy goals and problem constraints. Afterwards, a lot of articles dealing with fuzzy optimization problems were published.

In this article, we use the definition of the total order relation on L-fuzzy number space introduced by S. Saito and H. Ishi [9]. We define the twice H-differentiability of fuzzy-valued functions over $\mathbb{R}^n$. Using these concepts, we establish the first and second order necessary optimality conditions and second order sufficient optimality conditions for unconstrained multi-variable L-fuzzy optimization problems.

The paper is organized in 6 sections. In section 2, we give some basic definitions regarding fuzzy numbers and fuzzy arithmetic. Section 3 provides continuity and H-differentiability of fuzzy-valued functions defined on $\mathbb{R}^n$. We define a parametric total order relation on fuzzy number space and local optimum for fuzzy-valued functions in Section 4. Moreover, we prove first and second order optimality conditions for a local optimum. In Section 5, we provide some illustrative examples to justify the results. Finally, we conclude in Section 6 with a summary of the results established.

2. Fuzzy Numbers

Definition 1. [2, 8] Let $\mathbb{R}$ be the set of real numbers and $\tilde{a} : \mathbb{R} \to [0, 1]$ be a fuzzy set on $\mathbb{R}$. We say that $\tilde{a}$ is a fuzzy number if it satisfies the following properties:

(i) $\tilde{a}$ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $a(x_0) = 1$;
(ii) $\tilde{a}$ is convex, that is, $\tilde{a}(tx + (1-t)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$, whenever x, y $\in \mathbb{R}$ *and* $t \in [0, 1]$;

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(iii) $\tilde{a}(x)$ is upper semicontinuous on $\mathbb{R}$, that is, $\{x/\tilde{a}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0, 1]$;
(iv) $\tilde{a}_0 = cl\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$ forms a compact set.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. For all $\alpha \in (0, 1]$, α-level set $\tilde{a}_\alpha$ of any $\tilde{a} \in F(\mathbb{R})$ is defined as $\tilde{a}_\alpha = \{x \in \mathbb{R}/\tilde{a}(x) \geq \alpha\}$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$. By definition of fuzzy numbers, we can prove that, for any $\tilde{a} \in F(\mathbb{R})$ and for each $\alpha \in (0, 1]$, $\tilde{a}_\alpha$ is compact convex subset of $\mathbb{R}$, and we write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. $\tilde{a} \in F(\mathbb{R})$ can be recovered from its α-cuts by a well-known decomposition theorem, which states that

$$\tilde{a} = \cup_{\alpha \in [0,1]} \alpha \cdot \tilde{a}_\alpha$$

where union on the right-hand side is the standard fuzzy union.

Definition 2. [4, 7] For any $\tilde{a}, \tilde{b} \in F(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we define uniquely the addition $\tilde{a} \oplus \tilde{b}$, difference $\tilde{a} \ominus \tilde{b}$ and product $\lambda \odot \tilde{a}$ as fuzzy numbers by defining their α-cuts as follows:

$$\begin{aligned}
(\tilde{a} \oplus \tilde{b})_\alpha &= [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha] \\
(\tilde{a} \ominus \tilde{b})_\alpha &= [\tilde{a}^L_\alpha - \tilde{b}^U_\alpha, \tilde{a}^U_\alpha - \tilde{b}^L_\alpha] \\
(\lambda \odot \tilde{a})_\alpha &= [\lambda \cdot \tilde{a}^L_\alpha, \lambda \cdot \tilde{a}^U_\alpha], \ \forall \alpha \in [0, 1].
\end{aligned}$$

Definition 3. A fuzzy number $\tilde{a} \in F(\mathbb{R})$ is said to be a L-fuzzy number if its membership function $\mu_{\tilde{a}}$ is defined as

$$\mu_{\tilde{a}}(\xi) = \begin{cases} L(\frac{m-\xi}{l})_+ \ for \ \xi \leq m \\ L(\frac{\xi-m}{l})_+ \ for \ \xi > m \end{cases}$$

where $m \in \mathbb{R}$, $l > 0$. L is a mapping from $[0, \infty]$ into $[0, 1]$ and $L(\xi)_+ = \max(L(\xi), 0)$. The set of all L-fuzzy numbers on $\mathbb{R}$ is denoted by $F_L(\mathbb{R})$.

Definition 4. The membership function of a triangular fuzzy number $\tilde{a}$ is defined by

$$\zeta_{\tilde{a}}(r) = \begin{cases} \frac{(r-a^L)}{(a-a^L)} \ if \ a^L \leq r \leq a \\ \frac{(a^U-r)}{(a^U-a)} \ if \ a < r \leq a^U \\ 0 \ otherwise \end{cases}$$

which is denoted by $\tilde{a} = (a^L, a, a^U)$. Here $L(\xi)$ is given by

$$L(\xi) = \begin{cases} 1 - \frac{\xi}{(a-a^L)} \ if \ a^L \leq \xi \leq a \\ 1 - \frac{\xi}{(a^U-a)} \ if \ a < \xi \leq a^U \end{cases}$$

The α-level set of $\tilde{a}$ is

$$\tilde{a}_\alpha = [(1 - \alpha)a^L + \alpha a, (1 - \alpha)a^U + \alpha a].$$

Here, we can see that triangular fuzzy numbers are particular case of L-fuzzy numbers.

3. Differential Calculus of Fuzzy-valued Function

3.1. Continuity of Fuzzy-valued Function.

Definition 5. [7] We denote by K the set of all non-empty compact subsets of $\mathbb{R}^n$. The Hausdorff metric d_H on K as defined in [5], is given by

$$d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}||x-y||, \sup_{y\in B}\inf_{x\in A}||x-y||\}.$$

Then the metric d_F on $F(\mathbb{R})$ is defined as

$$d_F(\tilde{a},\tilde{b}) = \sup_{0\leq\alpha\leq 1}\{d_H(\tilde{a}_\alpha,\tilde{b}_\alpha)\},\ for\ all\ \tilde{a},\tilde{b}\in F(\mathbb{R}).$$

Since $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are closed bounded intervals in $\mathbb{R}$,

$$d_F(\tilde{a},\tilde{b}) = \sup_{0\leq\alpha\leq 1}[\max\{|\tilde{a}^L_\alpha - \tilde{b}^L_\alpha|, |\tilde{a}^U_\alpha - \tilde{b}^U_\alpha|\}].$$

Definition 6. [6] Let V be a real vector space and $F(\mathbb{R})$ be a fuzzy number space. Then a function $\tilde{f}: V \to F(\mathbb{R})$ is called fuzzy-valued function defined on V. Corresponding to such a function $\tilde{f}$ and $\alpha\in[0,1]$, we define two real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ on V as $\tilde{f}^L_\alpha(x) = (\tilde{f}(x))^L_\alpha$ and $\tilde{f}^U_\alpha(x) = (\tilde{f}(x))^U_\alpha$ for all $x\in V$.

Definition 7. [8] Let $\tilde{f}:\mathbb{R}^n\to F(\mathbb{R})$ be a fuzzy-valued function defined on $\mathbb{R}^n$. We say that $\tilde{f}$ is continuous at $c\in\mathbb{R}^n$ if for every $\epsilon>0$, there exists a $\delta>0$ such that, for all $x\in\mathbb{R}^n$, with $\|x-c\|<\delta$, we have $d_F(\tilde{f}(x),\tilde{f}(c))<\epsilon$. That is,

$$\lim_{x\to c}\tilde{f}(x) = \tilde{f}(c)$$

Proposition 1. [8] *Let $\tilde{f}:\mathbb{R}^n\to F(\mathbb{R})$ be a fuzzy-valued function on $\mathbb{R}^n$. If $\tilde{f}$ is continuous at $c\in\mathbb{R}^n$, then functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are continuous at c for all $\alpha\in[0,1]$.*

3.2. **H-differentiability of Fuzzy-valued Function on $\mathbb{R}$.** Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If there exists a fuzzy number $\tilde{c}$ such that $\tilde{c}\oplus\tilde{b}=\tilde{a}$. Then $\tilde{c}$ is called Hukuhara difference of $\tilde{a}$ and $\tilde{b}$ and is denoted by $\tilde{a}\ominus_H\tilde{b}$.

The following proposition is very useful for considering the differentiation of fuzzy-valued function.

Proposition 2. [8] *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If the Hukuhara difference $\tilde{c}=\tilde{a}\ominus_H\tilde{b}$ exists, then $\tilde{c}^L_\alpha=\tilde{a}^L_\alpha-\tilde{b}^L_\alpha$ and $\tilde{c}^U_\alpha=\tilde{a}^U_\alpha-\tilde{b}^U_\alpha$ for all $\alpha\in[0,1]$.*

Definition 8. [8]Let X be an open subset of $\mathbb{R}$. A fuzzy-valued function $\tilde{f}:X\to F(\mathbb{R})$ is said to be H-differentiable at x_0 if there exists a fuzzy number $D\tilde{f}(x_0)$ such that the limits

$$\lim_{h\to 0^+}\frac{1}{h}\odot[\tilde{f}(x_0+h)\ominus_H\tilde{f}(x_0)]\ and\ \lim_{h\to 0^+}\frac{1}{h}\odot[\tilde{f}(x_0)\ominus_H\tilde{f}(x_0-h)]$$

both exist and are equal to $D\tilde{f}(x_0)$. In this case, $D\tilde{f}(x_0)$ is called the H-derivative of $\tilde{f}$ at x_0.

Proposition 3. [8] *Let X be an open subset of $\mathbb{R}$. If a fuzzy-valued function $\tilde{f}:X\to F(\mathbb{R})$ is H-differentiable at x_0 with derivative $D\tilde{f}(x_0)$, then $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are differentiable at x_0 for all $\alpha\in[0,1]$. Moreover, we have $(D\tilde{f}(x_0))_\alpha = [D(\tilde{f}^L_\alpha)(x_0), D(\tilde{f}^U_\alpha)(x_0)]$.*

3.3. **H-differentiability of Fuzzy-valued Function on $\mathbb{R}^n$.**

Definition 9. [8] Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset X of $\mathbb{R}^n$ and let $\bar{x} = (\bar{x}_1, ..., \bar{x}_n) \in X$ be fixed.

(i) We say that $\tilde{f}$ has the i^{th} partial H-derivative $D_i\tilde{f}(\bar{x})$ at $\bar{x}$ if the fuzzy-valued function $\tilde{g}(x_i) = \tilde{f}(\bar{x}_1, .., \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, .., \bar{x}_n)$ is H-differentiable at $\bar{x}_i$ with H-derivative $D_i\tilde{f}(\bar{x})$. We also write $D_i\tilde{f}(\bar{x})$ as $(\partial\tilde{f}/\partial x_i)(\bar{x})$.

(ii) We say that $\tilde{f}$ is H-differentiable at $\bar{x}$ if all the partial H-derivatives $\partial\tilde{f}/\partial x_1, ..., \partial\tilde{f}/\partial x_n$ exists at $\bar{x}$ and all but except possibly one partial H-derivatives exist on some neighborhood of $\bar{x}$ and are continuous at $\bar{x}$ (in the sense of fuzzy-valued functions).

(iii) We say that $\tilde{f}$ is continuously H-differentiable at $\bar{x}$ if all of the partial H-derivatives $\partial\tilde{f}/\partial x_i, i = 1, ..., n$, exist on some neighborhood of $\bar{x}$ and are continuous at $\bar{x}$ (in the sense of fuzzy-valued functions).

Proposition 4. *Let X be an open subset of $\mathbb{R}^n$. If a fuzzy-valued function $\tilde{f}$: $X \to F(\mathbb{R})$ is H-differentiable at $\bar{x} \in X$. Then $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also differentiable at $\bar{x} \in X$, for all $\alpha \in [0, 1]$. Moreover, $(D_i\tilde{f}(\bar{x}))_\alpha = [D_i(\tilde{f}^L_\alpha)(\bar{x}), D_i(\tilde{f}^U_\alpha)(\bar{x})]$, i = 1,...,n.*

Proof. The result follows from Proposition 1 and 3. □

Proposition 5. *Let X be an open subset of $\mathbb{R}^n$. If a fuzzy-valued function $\tilde{f}$: $X \to F(\mathbb{R})$ is continuously H-differentiable at $\bar{x} \in X$. Then $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also continuously differentiable at $\bar{x}$, for all $\alpha \in [0, 1]$.*

Let $\tilde{f}$ be H-differentiable at $\bar{x}$. Then the gradient of $\tilde{f}$ at $\bar{x}$ is denoted by

$$\nabla\tilde{f}(\bar{x}) = (D_1\tilde{f}(\bar{x}), ..., D_n\tilde{f}(\bar{x})),$$

and it defines a fuzzy-valued function from X to $F^n(\mathbb{R}) = F(\mathbb{R}) \times .... \times F(\mathbb{R})$ (n times), where each $D_i\tilde{f}(\bar{x})$ is a fuzzy number for i = 1,...,n. The α-level set of $\nabla\tilde{f}(\bar{x})$ is defined and denoted by

$$(\nabla\tilde{f}(\bar{x}))_\alpha = ((D_1\tilde{f}(\bar{x}))_\alpha, ..., (D_n\tilde{f}(\bar{x}))_\alpha),$$

where

$$(D_i\tilde{f}(\bar{x}))_\alpha = [D_i(\tilde{f}^L_\alpha)(\bar{x}), D_i(\tilde{f}^U_\alpha)(\bar{x})],$$

i = 1,...,n.

Definition 10. Let $\tilde{f} : X \to F(\mathbb{R}), X \subset \mathbb{R}^n$ be a fuzzy-valued function. Suppose now that there is $\bar{x} \in X$ such that gradient of $\tilde{f}$, $\nabla\tilde{f}$, is itself H-differentiable at $\bar{x}$, that is, for each i, the function $D_i\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at $\bar{x}$. Denote the H-partial derivative of $D_i\tilde{f}$ in the direction of $\bar{e}_j$ at $\bar{x}$ by

$$D^2_{ij}\tilde{f} \ or \ \frac{\partial^2\tilde{f}(\bar{x})}{\partial x_i \partial x_j}, \ if \ i \neq j,$$

and

$$D^2_{ii}\tilde{f} \ or \ \frac{\partial^2\tilde{f}(\bar{x})}{\partial x_i^2}, \ if \ i = j.$$

Then we say that $\tilde{f}$ is twice H-differentiable at $\bar{x}$, with second H-derivative $\nabla^2\tilde{f}(\bar{x})$ which is denoted by

$$\nabla^2\tilde{f}(\bar{x}) = \begin{pmatrix} \frac{\partial^2\tilde{f}(\bar{x})}{\partial x_1^2} & \dots & \frac{\partial^2\tilde{f}(\bar{x})}{\partial x_1 \partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial^2\tilde{f}(\bar{x})}{\partial x_n \partial x_1} & \dots & \frac{\partial^2\tilde{f}(\bar{x})}{\partial x_n^2} \end{pmatrix}$$

where $\frac{\partial^2\tilde{f}(\bar{x})}{\partial x_i \partial x_j} \in F(\mathbb{R})$, i,j = 1,...,n.

If $\tilde{f}$ is twice H-differentiable at each $\bar{x}$ in X, we say that $\tilde{f}$ is twice H-differentiable on X, and if for each i, j = 1,...,n, the cross-partial derivative $\frac{\partial^2\tilde{f}}{\partial x_i \partial x_j}$ is continuous function from X to $F(\mathbb{R})$, we say that $\tilde{f}$ is twice continuously H-differentiable on X.

The α-level set of $\nabla^2\tilde{f}(\bar{x})$ is defined and denoted in matrix notation as

$$(\nabla^2\tilde{f}(\bar{x}))_\alpha = ((D^2_{ij}\tilde{f}(\bar{x}))_\alpha)$$

i,j = 1,...,n and $\alpha \in [0,1]$, where $(D^2_{ij}\tilde{f}(\bar{x}))_\alpha$ denotes α-cut of $(D^2_{ij}\tilde{f}(\bar{x}))$.

Proposition 6. *Let $\tilde{f} : X \subseteq \mathbb{R}^n \to F(\mathbb{R})$ is differentiable with derivative $\nabla\tilde{f}$ on X and let each $D_i\tilde{f} : X \to F(\mathbb{R})$, i = 1,...,n, is also differentiable at $\bar{x}$ with derivative $D^2_{ij}\tilde{f}(\bar{x})$, i , j = 1,...,n. Then $D_i\tilde{f}^L_\alpha$ and $D_i\tilde{f}^U_\alpha$ are also differentiable at $\bar{x}$, for all $\alpha \in [0,1]$. Also, we have $(D^2_{ij}\tilde{f}(\bar{x}))_\alpha = [D^2_{ij}(\tilde{f}^L_\alpha)(\bar{x}), D^2_{ij}(\tilde{f}^U_\alpha)(\bar{x})]$, i,j = 1,...,n.*

Proof. Follows by Proposition 3.4. □

4. Optimality Conditions

4.1. **Total order relation on $F_L(\mathbb{R})$.** Let $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ be a fuzzy-valued function, where $F_L(\mathbb{R})$ is the set of L-fuzzy numbers. Inspired by [9], we define here a total order relation on L-fuzzy numbers as follows

Definition 11. For any $\tilde{a}, \tilde{b} \in F_L(\mathbb{R})$, we say that $\tilde{a} \preceq_\lambda \tilde{b}$, where "$\preceq_\lambda$" is a parametric order relation on $F_L(\mathbb{R})$, for $0 \le \lambda \le 1$ if only one of the following inequalities hold:

(i) $\lambda[\tilde{a}^L_1 - \tilde{a}^L_0] + \tilde{a}^L_1 < \lambda[\tilde{b}^L_1 - \tilde{b}^L_0] + \tilde{b}^L_1$ for $\tilde{a}^L_1 - \tilde{a}^L_0 < \tilde{b}^L_1 - \tilde{b}^L_0$

(ii) $\lambda[\tilde{a}^L_1 - \tilde{a}^L_0] + \tilde{a}^L_1 \le \lambda[\tilde{b}^L_1 - \tilde{b}^L_0] + \tilde{b}^L_1$ for $\tilde{a}^L_1 - \tilde{a}^L_0 \ge \tilde{b}^L_1 - \tilde{b}^L_0$

It can be easily proved that "$\preceq_\lambda$" for any fixed $\lambda \in [0,1]$ is a total order relation on $F_L(\mathbb{R})$. $\tilde{a} \succeq_\lambda \tilde{b}$ is defined by $\tilde{b} \preceq_\lambda \tilde{a}$.

We write $\tilde{a} \prec_\lambda \tilde{b}$ if and only if $\tilde{a} \not\succeq_\lambda \tilde{b}$. That is, $\tilde{a} \prec_\lambda \tilde{b}$ if and only if the following inequalities fail simultaneously or hold simultaneously

(i) $\lambda[\tilde{b}^L_1 - \tilde{b}^L_0] + \tilde{b}^L_1 < \lambda[\tilde{a}^L_1 - \tilde{a}^L_0] + \tilde{a}^L_1$ for $\tilde{b}^L_1 - \tilde{b}^L_0 < \tilde{a}^L_1 - \tilde{a}^L_0$

(ii) $\lambda[\tilde{b}^L_1 - \tilde{b}^L_0] + \tilde{b}^L_1 \le \lambda[\tilde{a}^L_1 - \tilde{a}^L_0] + \tilde{a}^L_1$ for $\tilde{b}^L_1 - \tilde{b}^L_0 \ge \tilde{a}^L_1 - \tilde{a}^L_0$

Also $\tilde{a} \succ_\lambda \tilde{b}$ is defined by $\tilde{b} \prec_\lambda \tilde{a}$.

4.2. **First-Order Condition.** Consider unconstrained L-fuzzy optimization problem

$$Minimize\ \tilde{f}(\bar{x}),\ \bar{x} \in X$$

where $\tilde{f} : \mathbb{R}^n \to F_L(\mathbb{R})$ is a fuzzy-valued function and $X \subseteq \mathbb{R}^n$. First we define local minimum (maximum) and strict local minimum (maximum) of a fuzzy-valued function

Definition 12. Let $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ and $\lambda \in [0,1]$ be fixed.

(1) A point $\bar{x}^* \in X$ is a local minimum (maximum) of $\tilde{f}$ with respect to parametric total order relation "$\preceq_\lambda$" , if there exists $r > 0$ such that $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x})$ ($\tilde{f}(\bar{x}^*) \succeq_\lambda \tilde{f}(\bar{x})$), for all $\bar{x} \in X \cap B(\bar{x}^*; r)$.

(2) A point $\bar{x}^* \in X$ is called a strict local minimum (maximum) of $\tilde{f}$ with respect to parametric total order relation "$\preceq_\lambda$", if there exists $r > 0$ such that $\tilde{f}(\bar{x}^*) \prec_\lambda \tilde{f}(\bar{x})$ ($\tilde{f}(\bar{x}^*) \succ_\lambda \tilde{f}(\bar{x})$), for all $\bar{x} \in X \cap B(\bar{x}^*; r)$.

We present now first-order necessary condition for optimality of a multi-variable fuzzy-valued function.

Theorem 1. ***(FONC)*** *Suppose $\bar{x}^* \in intX = \{\bar{x} \in X$ / there exists $r > 0$ such that $B(\bar{x}^*; r) \subset X\} \subseteq \mathbb{R}^n$ be a local minimizer of $\tilde{f} : X \to F_L(\mathbb{R})$ with respect to parametric total order relation "$\preceq_\lambda$". Suppose also that $\tilde{f}$ is H-differentiable at $\bar{x}^*$. Then $\lambda[\nabla\tilde{f}_1^L(\bar{x}^*) - \nabla\tilde{f}_0^L(\bar{x}^*)] + \nabla\tilde{f}_1^L(\bar{x}^*) = 0$.*

Proof. Since $\bar{x}^* \in intX$ a local minimum of $\tilde{f}$ on X, by definition, we have $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x})$ for all $\bar{x} \in X \cap B(\bar{x}^*; r)$.

That is, only one of the following inequalities hold:

(i) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$
for $\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*) < \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$

(ii) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) \leq \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$
for $\tilde{f}_1^L(x^*) - \tilde{f}_0^L(x^*) \geq \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$.

Therefore, for any $\bar{x} \in X \cap B(\bar{x}^*; r)$ we have either

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}).$$

or

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) \leq \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}).$$

Let $\bar{e}_i = [0, ..., 1, ...0]^T$ be a unit vector 1 in the i^{th} location. Then $(\bar{x}^* + h\bar{e}_i)$ with $h > 0$ will represent a purturbtation of magnitude h in $\bar{x}^*$ in the direction $\bar{e}_i$.

Let $\bar{x} = \bar{x}^* + h\bar{e}_i$, where $h < r$, then we have

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \ or\ \leq\ \lambda[\tilde{f}_1^L(\bar{x}^* + h) - \tilde{f}_0^L(\bar{x}^* + h)] + \tilde{f}_1^L(\bar{x}^* + h)$$

Similarly, for $\bar{x} = \bar{x}^* - h\bar{e}_i$, we have

$$\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \ or\ \leq\ \lambda[\tilde{f}_1^L(\bar{x}^* - h) - \tilde{f}_0^L(\bar{x}^* - h)] + \tilde{f}_1^L(\bar{x}^* - h)$$

for sufficiently small h. That is,

(4.1)
$$\lambda\Big(\tilde{f}_1^L(\bar{x}^* + h) - \tilde{f}_1^L(\bar{x}^*)\Big) - \lambda\Big(\tilde{f}_0^L(\bar{x}^* + h) - \tilde{f}_0^L(\bar{x}^*)\Big) + \tilde{f}_1^L(\bar{x}^* + h) - \tilde{f}_1^L(\bar{x}^*) > \ or\ \geq 0.$$

(4.2)
$$\lambda\Big(\tilde{f}_1^L(\bar{x}^* - h) - \tilde{f}_1^L(\bar{x}^*)\Big) - \lambda\Big(\tilde{f}_0^L(\bar{x}^* - h) - \tilde{f}_0^L(\bar{x}^*)\Big) + \tilde{f}_1^L(\bar{x}^* - h) - \tilde{f}_1^L(\bar{x}^*) > \ or\ \geq 0.$$

Since $\tilde{f}$ is H-differentiable at $\bar{x}^*$, by Proposition 3.4, $\tilde{f}_\alpha^L$ is also differentiable at $\bar{x}^*$ for all $\alpha \in [0,1]$. Dividing the inequalities (4.1) and (4.2) by h and $-h$ respectively and taking limit as $h \to 0$, we get

$$\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) \geq 0$$

$$\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) \leq 0$$

which gives

$$\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) = 0.$$

□

4.3. **Second-Order Conditions.** In this section, first we present the second-order necessary conditions for optimality of a fuzzy-valued function defined on $\mathbb{R}^n$.

Theorem 2. ***(SONC)*** *Suppose $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ be a continuously H-differentiable fuzzy-valued function, and $\bar{x}^*$ in X is a point in the interior of X.*

(1) *If $\tilde{f}$ has a local minimum at $\bar{x}^*$, then $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is positive semidefinite.*
(2) *If $\tilde{f}$ has a local maximum at $\bar{x}^*$, then $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is negative semidefinite.*

We adopt a two step procedure to prove this theorem. We first prove this result for the case where n = 1 i.e., ($X \subseteq \mathbb{R}$) and then we use this result to prove the general case.

Proof. Case 1: n = 1

When $n = 1$, $\tilde{f} : X \subseteq \mathbb{R} \to F_L(\mathbb{R})$ and $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is real number. We have to prove that

$$\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*) \geq 0.$$

Since $\tilde{f}$ has a local minimum at $\bar{x}^*$, by definition, we have $\tilde{f}(\bar{x}^*) \preceq_\lambda \tilde{f}(\bar{x})$ for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and $r > 0$. That is, only one of the following inequalities hold:

(i) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) < \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$ for $\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*) < \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$
(ii) $\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*) \leq \lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x})$ for $\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*) \geq \tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})$

for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and $0 \leq \lambda \leq 1$.

Consider Taylor's series expansion of $\tilde{f}_\alpha^L$ at $\bar{x}^*$ for sufficiently small h such that $\bar{x}^* + h \in B(\bar{x}^*; r)$ and

$$\tilde{f}_\alpha^L(\bar{x}^* + h) = \tilde{f}_\alpha^L(\bar{x}^*) + hD\tilde{f}_\alpha^L(\bar{x}^*) + \frac{1}{2}h^2 D^2 \tilde{f}_\alpha^L(\bar{x}^*) + O(h^3)$$

Using this,

$$\Big\{\lambda[\tilde{f}_1^L(\bar{x}^*+h)-\tilde{f}_0^L(\bar{x}^*+h)]+\tilde{f}_1^L(\bar{x}^*+h)\Big\} = \Big\{\lambda[\tilde{f}_1^L(\bar{x}^*)-\tilde{f}_0^L(\bar{x}^*)]+\tilde{f}_1^L(\bar{x}^*)\Big\}+ +h\Big\{\lambda[D\tilde{f}_1^L(\bar{x}^*)-D\tilde{f}_0^L(\bar{x}^*)]+ D\tilde{f}_1^L(\bar{x}^*)\Big\}+ \frac{h^2}{2}\Big\{\lambda[D^2\tilde{f}_1^L(\bar{x}^*)-D^2\tilde{f}_0^L(\bar{x}^*)]+ D^2\tilde{f}_1^L(\bar{x}^*)\Big\} +O(h^3)$$

At local minimum, $\lambda[D\tilde{f}_1^L(\bar{x}^*)-D\tilde{f}_0^L(\bar{x}^*)]+D\tilde{f}_1^L(\bar{x}^*)=0$. Upon choosing h sufficiently small, we can ensure that the term

$$\frac{h^2}{2}\Big\{\lambda[D^2\tilde{f}_1^L(\bar{x}^*)-D^2\tilde{f}_0^L(\bar{x}^*)]+D^2\tilde{f}_1^L(\bar{x}^*)\Big\}$$

dominates the remainder term $O(h^3)$. Thus at a local minimum, we have

$$\lambda[D^2\tilde{f}_1^L(\bar{x}^*)-D^2\tilde{f}_0^L(\bar{x}^*)]+D^2\tilde{f}_1^L(\bar{x}^*)\geq 0.$$

Case 2 : $n>1$

We prove part 1. Let $\bar{x}^*$ be a local minimum of $\tilde{f}$ on X. We have to show that for any $z\in\mathbb{R}^n$, $z\neq 0$, we have $z^{'}Az\geq 0$, where

$$A=\lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*)-\nabla^2\tilde{f}_0^L(\bar{x}^*)]+\nabla^2\tilde{f}_1^L(\bar{x}^*).$$

Pick any $z\in\mathbb{R}^n$, define the fuzzy-valued function $\tilde{g}:\mathbb{R}\to F(\mathbb{R})$ by $\tilde{g}(t)=\tilde{f}(\bar{x}^*+tz)$.

Note that $\tilde{g}(0)=\tilde{f}(\bar{x}^*)$. For $|t|$ sufficiently small, $\tilde{f}(\bar{x}^*)\preceq_\lambda \tilde{f}(\bar{x}^*+tz)$, since $\tilde{f}(\bar{x})$ has a local minimum at $\bar{x}^*$.

It follows that there exists a $\epsilon>0$ such that $\tilde{g}(0)\preceq_\lambda\tilde{g}(t)$ for all $t\in(-\epsilon,\epsilon)$. That is, 0 is a local minimum of $\tilde{g}$.

By case 1, therefore we must have

$$\lambda[D^2\tilde{g}_1^L(0)-D^2\tilde{g}_0^L(0)]+D^2\tilde{g}_1^L(0)\geq 0.$$

On the other hand, it follows from the definition of $\tilde{g}$, that $\tilde{g}$ is twice continuously H-differentiable, as $\tilde{g}(t)=\tilde{f}(x^*+tz)$ and

$$\lambda[D^2\tilde{g}_1^L(0)-D^2\tilde{g}_0^L(0)]+D^2\tilde{g}_1^L(0)=z^{'}Az$$

where $A=\lambda[\nabla^2\tilde{f}_1^L(\bar{x}^*)-\nabla^2\tilde{f}_0^L(\bar{x}^*)]+\nabla^2\tilde{f}_1^L(\bar{x}^*)$, so that

$$z^{'}Az=\lambda[D^2\tilde{g}_1^L(0)-D^2\tilde{g}_0^L(0)]+D^2\tilde{g}_1^L(0)\geq 0,$$

as desired. This completes the proof of Part 1. Part 2 is proved similarly. □

Now we prove the second-order sufficient conditions for $\bar{x}^*$ to be a strict local minimizer (maximizer) of a fuzzy-valued function defined on $\mathbb{R}^n$.

Theorem 3. ***(SOSC)*** *Suppose $\tilde{f} : X \subseteq \mathbb{R}^n \to F_L(\mathbb{R})$ is a twice continuously H-differentiable function.*

(1) *If $\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) = 0$ and $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is positive definite, then $\bar{x}^*$ is a strict local minimum of $\tilde{f}$ on X.*

(2) *If $\lambda[\nabla \tilde{f}_1^L(\bar{x}^*) - \nabla \tilde{f}_0^L(\bar{x}^*)] + \nabla \tilde{f}_1^L(\bar{x}^*) = 0$ and $\lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$ is negative definite, then $\bar{x}^*$ is a strict local maximum of $\tilde{f}$ on X.*

Proof. We prove Part 1, Part 2 is proved similarly. Here we have to prove that $\bar{x}^*$ is a strict local minimum of $\tilde{f}$ on X. That is, by definition

$$\tilde{f}(\bar{x}^*) \prec_\lambda \tilde{f}(\bar{x})$$

for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and for fixed $\lambda \in [0, 1]$.

That means, we have show that the following inequalities fail simultaneously or hold simultaneously.

(i) $\lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}) < \lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)$
for $\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x}) < \tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)$

(ii) $\lambda[\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x})] + \tilde{f}_1^L(\bar{x}) \leq \lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)$
for $\tilde{f}_1^L(\bar{x}) - \tilde{f}_0^L(\bar{x}) \geq \tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)$

for all $\bar{x} \in X \cap B(\bar{x}^*; r)$ and for fixed $\lambda \in [0, 1]$.

Let $\tilde{H}^L(\bar{x}^*) = \lambda[\nabla^2 \tilde{f}_1^L(\bar{x}^*) - \nabla^2 \tilde{f}_0^L(\bar{x}^*)] + \nabla^2 \tilde{f}_1^L(\bar{x}^*)$. Using assumption 2, and Rayleigh's inequality (Ref. [3], pp. 34) it follows that if $\bar{d} \neq 0$ then

$$\lambda_{min}(\tilde{H}^L(\bar{x}^*))\|\bar{d}\|^2 \leq \bar{d}^T \tilde{H}^L(\bar{x}^*)\bar{d}$$

where $\lambda_{min}(\tilde{H}^L(\bar{x}^*))$ is the smallest eigen value of $\tilde{H}^L(\bar{x}^*)$. By Taylor's theorem and assumption 1,

$$\begin{aligned}
\Big\{\lambda[\tilde{f}_1^L(\bar{x}^* + d) - \tilde{f}_0^L(\bar{x}^* + d)] + \tilde{f}_1^L(\bar{x}^* + d)\Big\} \quad &- \quad \Big\{\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)\Big\} \\
&= \quad \frac{1}{2}\bar{d}^T \tilde{H}^L(\bar{x}^*)\bar{d} + O(\|\bar{d}\|^2) \\
&\geq \quad \frac{\lambda_{min}(\tilde{H}^L(\bar{x}^*))}{2}\|\bar{d}\|^2 + O(\|\bar{d}\|^2)
\end{aligned}$$

Hence for all $\bar{d}$ such that $\|\bar{d}\|$ is sufficiently small,

$$\{\lambda[\tilde{f}_1^L(\bar{x}^* + d) - \tilde{f}_0^L(\bar{x}^* + d)] + \tilde{f}_1^L(\bar{x}^* + d)\} > \{\lambda[\tilde{f}_1^L(\bar{x}^*) - \tilde{f}_0^L(\bar{x}^*)] + \tilde{f}_1^L(\bar{x}^*)\}.$$

This inequality fails the above two inqualities (i) and (ii) simultaneously. Therefore we say that $\tilde{f}$ has strict local minimizer at $\bar{x}^*$ with respect to the total order relation " $\preceq_\lambda$ ". □

5. Illustrative Examples

Example 1.

$$\textit{Minimize } \tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^2) \oplus (\widetilde{0.5} \odot x_2^2) \oplus (\tilde{3} \odot x_2) \oplus \widetilde{4.5}, \ x_1, \ x_2 \in \mathbb{R}$$

where $\tilde{1} = (0, 1, 2)$, $\widetilde{0.5} = (0.4, 0.5, 0.6)$, $\tilde{3} = (2, 3, 4)$ and $\widetilde{4.5} = (3.5, 4.5, 5.5)$ are triangular fuzzy numbers defined on $\mathbb{R}$ and with respect to total order relation " $\preceq_\lambda$ " for some fixed $\lambda \in [0, 1]$.

Here $\tilde{f}^L_\alpha(x_1, x_2) = \alpha x_1^2 + (0.4 + \alpha 0.1)x_2^2 + (2 + \alpha)x_2 + (3.5 + \alpha)$,

$$\nabla \tilde{f}^L_\alpha = \begin{pmatrix} 2\alpha x_1 \\ 2(0.4 + 0.1\alpha)x_2 + (2 + \alpha) \end{pmatrix}.$$

By first order necessary condition $\lambda[\nabla \tilde{f}^L_1(\bar{x}) - \nabla \tilde{f}^L_0(\bar{x})] + \nabla \tilde{f}^L_1(\bar{x}) = 0$.
That is

$$\lambda 2x_1 + 2x_1 = 0$$
$$\lambda(0.2x_2 + 1) + x_2 + 3 = 0$$

Solving these equations, we get parametric solution

$$\bar{x}^* = \Big(0, -\frac{(\lambda + 3)}{0.2\lambda + 1}\Big)$$

Now

$$\lambda[\nabla^2 \tilde{f}^L_1(\bar{x}) - \nabla^2 \tilde{f}^L_0(\bar{x})] + \nabla^2 \tilde{f}^L_1(\bar{x}) = \begin{pmatrix} 2\lambda + 2 & 0 \\ 0 & 0.2\lambda + 1 \end{pmatrix}$$

where

$$\nabla^2 \tilde{f}^L_\alpha(\bar{x}) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2(0.4 + 0.1\alpha) \end{pmatrix}.$$

Since this matrix is positive definite for all $\lambda \in [0, 1]$, *the point* $\bar{x}^* = \Big(0, -\frac{(\lambda+3)}{0.2\lambda+1}\Big)$ *satisfies the* ***FONC***, ***SONC*** *and* ***SOSC***. *So it is a strict local minimizer of given fuzzy-valued function.*

Example 2.

$$Minimize\ \tilde{f}(x_1, x_2) = (\tilde{1} \odot x_1^2) \oplus ((\widetilde{-1}) \odot x_2^2),\ x_1, x_2 \in \mathbb{R}$$

where $\tilde{1} = (0, 1, 2)$ *and* $(\widetilde{-1}) = (-2, -1, 0)$ *are triangular fuzzy numbers defined on* $\mathbb{R}$ *and with respect to total order relation "* $\preceq_\lambda$ *" for some fixed* $\lambda \in [0, 1]$.

Here $\tilde{f}^L_\alpha(x_1, x_2) = \alpha x_1^2 + (-2 + \alpha)x_2^2$,

$$\nabla \tilde{f}^L_\alpha = \begin{pmatrix} 2\alpha x_1 \\ 2(-2 + \alpha)x_2 \end{pmatrix}.$$

By first order necessary condition $\lambda[\nabla \tilde{f}^L_1(\bar{x}) - \nabla \tilde{f}^L_0(\bar{x})] + \nabla \tilde{f}^L_1(\bar{x}) = 0$.
That is

$$\lambda 2x_1 + 2x_1 = 0$$
$$\lambda 2x_2 - 2x_2 = 0.$$

Solving these equations, we get the solution

$$\bar{x}^* = (0, 0).$$

We evaluate

$$\lambda[\nabla^2 \tilde{f}^L_1(\bar{x}) - \nabla^2 \tilde{f}^L_0(\bar{x})] + \nabla^2 \tilde{f}^L_1(\bar{x}) = \begin{pmatrix} 2\lambda + 2 & 0 \\ 0 & 2\lambda - 2 \end{pmatrix}$$

where

$$\nabla^2 \tilde{f}^L_\alpha(\bar{x}) = \begin{pmatrix} 2\alpha & 0 \\ 0 & 2(-2 + \alpha) \end{pmatrix}.$$

The point $\bar{x}^* = (0,0)$ *satisfies the* ***FONC*** *but* ***SONC*** *is not satisfied, since* $2\lambda+2 > 0$ *but* $(2\lambda+2)(2\lambda-2) \leq 0$ *for* $\lambda \in [0,1)$. *Therefore* $\bar{x}^* = (0,0)$ *is not optimum point for given fuzzy-valued function, if* $\lambda \in [0,1)$.

6. Conclusion

In this article we have used the concept of L-fuzzy numbers and a total order relation "$\preceq_\lambda$" on space of L-fuzzy numbers as introduced in [9]. By considering the optimization problems with respect to this total order relation, we have derived the first and second order neccessary conditions as well as second order sufficient conditions for optimality of a fuzzy-valued function defined on $\mathbb{R}^n$. We have also given two illustrative examples to justify the conditions.

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