Sutra: International Journal of Mathematical Science Education

Vol. 4 No. 1, pp. 1 - 16, 2011

Necessary and Sufficient Optimality Conditions for Nonlinear Fuzzy Optimization Problem

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In this paper we derive the necessary and sufficient Kuhn-Tucker like optimality conditions for nonlinear fuzzy optimization problems with fuzzy valued objective function and fuzzy-valued constraints using the concept of convexity and H-diffrentiability of fuzzy-valued functions.

Keywords: Fuzzy numbers · Hukuhara differentiability · Kuhn-Tucker optimality conditions

1 Introduction

Classical optimization techniques have been successfully applied for years. In real process optimization, there exist different types of uncertainties in the system. Zimmermann [16] pointed out various kinds of uncertainties that can be categorized as stochastic uncertainty and fuzziness. The optimization under a fuzzy environment or which involve fuzziness is called fuzzy optimization.

Bellman and Zadeh in 1970 [1] proposed the concept of fuzzy decision and the decision model under fuzzy environments. After that, various approaches to fuzzy linear and nonlinear optimization, have been developed over the years by researchers.

The nondominated solution of a nonlinear optimization problem with fuzzy-valued objective function was proposed by Wu [10]. Using the concept of continuous differentiability of fuzzy-valued functions, he derived the sufficient optimality conditions for obtaining the nondominated solution of fuzzy optimization problem having fuzzy- valued objective function with real constraints. However, the fuzzy optimization problem having fuzzy-valued constraints can not be solved by using the results of Wu [10]. In this article, we establish Kuhn-Tucker like both necessary and sufficient optimality conditions for obtaining the nondominated solution of a nonlinear fuzzy optimization

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problem with fuzzy-valued objective function and fuzzy-valued constraints.
In Section 2, we introduce definition of fuzzy number, basic properites and arithmetics of fuzzy numbers. In Section 3, we consider the differential calculus of fuzzy-valued functions defined on $\mathbb{R}$ and $\mathbb{R}^n$ using hukuhara differentiability of fuzzy-valued functions. In Section 4, we provide nondominated solution of unconstrained fuzzy optimization problems by proving the first and second order optimality conditions. In Section 5, we provide nondominated solution of nonlinear constrained fuzzy optimization problems by proving the Kuhn-Tucker like optimality conditions for the same. And at last we conclude in Section 6.

2 Preliminaries

Definition 1 [6] Let $\mathbb{R}$ be the set of real numbers and $\tilde{a} : \mathbb{R} \rightarrow [0, 1]$ be a fuzzy set. We say that $\tilde{a}$ is a fuzzy number if it satisfies the following properties:

(i) $\tilde{a}$ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $\tilde{a}(x_0) = 1$;
(ii) $\tilde{a}$ is fuzzy convex, that is, $\tilde{a}(tx + (1 - t)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$, whenever x, y $\in \mathbb{R}$ $\ and \ \ t \in [0, 1]$;
(iii) $\tilde{a}(x)$ is upper semicontinuous on $\mathbb{R}$, that is, $\{x/\tilde{a}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0, 1]$;
(iv) $cl\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$ forms a compact set.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. For all $\alpha \in (0, 1]$, α-level set $\tilde{a}_\alpha$ of any $\tilde{a} \in F(\mathbb{R})$ is defined as $\tilde{a}_\alpha = \{x \in \mathbb{R}/\tilde{a}(x) \geq \alpha\}$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$. By definition of fuzzy numbers, we can prove that, for any $\tilde{a} \in F(\mathbb{R})$ and for each $\alpha \in (0, 1]$, $\tilde{a}_\alpha$ is compact convex subset of $\mathbb{R}$, and we write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. $\tilde{a} \in F(\mathbb{R})$ can be recovered from its α-cuts by a well-known decomposition theorem (ref. [7]), which states that $\tilde{a} = \cup_{\alpha\in[0,1]}\alpha \cdot \tilde{a}_\alpha$ where union on the right-hand side is the standard fuzzy union.

Definition 2 [15] According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space $F(\mathbb{R})$ by their α-cuts are as follows:

$$(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha]$$
$$(\lambda \odot \tilde{a})_\alpha = [\lambda \cdot \tilde{a}^L_\alpha, \lambda \cdot \tilde{a}^U_\alpha],$$

where $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $\alpha \in [0, 1]$.

Definition 3 [9] Let $A, B \subseteq \mathbb{R}^n$. The Hausdorff metric d_H is defined by

$$d_H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} ||x - y||, \sup_{y\in B} \inf_{x\in A} ||x - y||\}.$$

Then the **metric** d_F on $F(\mathbb{R})$ is defined as

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0\leq\alpha\leq1} \{d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)\},$$

for all $\tilde{a}, \tilde{b} \in F(\mathbb{R})$. Since $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are closed bounded intervals in $\mathbb{R}$,

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0\leq\alpha\leq1} \max\{|\tilde{a}^L_\alpha - \tilde{b}^L_\alpha|, |\tilde{a}^U_\alpha - \tilde{b}^U_\alpha|\}.$$

We need the following proposition.

Proposition 1 *[3] For $\tilde{a} \in F(\mathbb{R})$, we have*

(i) $\tilde{a}^L_\alpha$ is bounded left continuous nondecreasing function on (0,1];
(ii) $\tilde{a}^U_\alpha$ is bounded left continuous nonincreasing function on (0,1];
(iii) $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$ are right continuous at $\alpha = 0$;
(iv) $\tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha$.

Moreover, if the pair of functions $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$ satisfy the conditions (i)-(iv), then there exists a unique $\tilde{a} \in F(\mathbb{R})$ such that $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$, for each $\alpha \in [0,1]$.

We define here a partial order relation on fuzzy number space.

Definition 4 For $\tilde{a}$ and $\tilde{b}$ in $F(\mathbb{R})$ and $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ and $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ are two closed intervals in $\mathbb{R}$, for all $\alpha \in [0,1]$, we define

(i) $\tilde{a} \preceq \tilde{b}$ if and only if $\tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha$ and $\tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha$ for all $\alpha \in [0,1]$;
(ii) $\tilde{a} \prec \tilde{b}$ if and only if

$$\begin{cases} \tilde{a}^L_\alpha < \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha \leq \tilde{b}^U_\alpha \end{cases} \text{or} \begin{cases} \tilde{a}^L_\alpha \leq \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha < \tilde{b}^U_\alpha \end{cases} \text{or} \begin{cases} \tilde{a}^L_\alpha < \tilde{b}^L_\alpha \\ \tilde{a}^U_\alpha < \tilde{b}^U_\alpha \end{cases}$$

for all $\alpha \in [0,1]$.

"$\preceq$" is partial order relation on fuzzy number space.

Definition 5 [14] The membership function of a triangular fuzzy number $\tilde{a}$ is defined by

$$\zeta_{\tilde{a}}(r) = \begin{cases} \frac{(r-a^L)}{(a-a^L)} & if \quad a^L \leq r \leq a \\ \frac{(a^U-r)}{(a^U-a)} & if \quad a < r \leq a^U \\ 0 & otherwise \end{cases}$$

which is denoted by $\tilde{a} = (a^L, a, a^U)$. The α-level set of $\tilde{a}$ is then

$$\tilde{a}_\alpha = [(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a].$$

3 Differential calculus of fuzzy-valued function

3.1 Continuity of fuzzy-valued function

Definition 6 [8] Let V be a real vector space and $F(\mathbb{R})$ be a fuzzy number space. Then a function $\tilde{f} : V \to F(\mathbb{R})$ is called fuzzy-valued function defined on V.
Corresponding to such a function $\tilde{f}$ and $\alpha \in [0,1]$, we define two real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ on V as $\tilde{f}^L_\alpha(x) = (\tilde{f}(x))^L_\alpha$ and $\tilde{f}^U_\alpha(x) = (\tilde{f}(x))^U_\alpha$ for all $x \in V$.

Definition 7 [4] Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. We say that $\tilde{f}$ is continuous at $c \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists a $\delta = \delta(c, \epsilon) > 0$ such that

$$d_F(\tilde{f}(x), \tilde{f}(c)) < \epsilon$$

for all $x \in \mathbb{R}^n$ with $\|x - c\| < \delta$. That is,

$$\lim_{x \to c} \tilde{f}(x) = \tilde{f}(c).$$

We prove the following proposition.

Proposition 2 *Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. If $\tilde{f}$ is continuous at $c \in \mathbb{R}^n$, then functions $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ are continuous at c, for all $\alpha \in [0,1]$.*

Proof The result follows using the definitions of continuity of fuzzy-valued function $\tilde{f}$ and metric on fuzzy numbers. □

3.2 H-differentiability of fuzzy-valued function on $\mathbb{R}$

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If there exists a fuzzy number $\tilde{c}$ such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$. Then $\tilde{c}$ is called **Hukuhara difference** of $\tilde{a}$ and $\tilde{b}$ and is denoted by $\tilde{a} \ominus_H \tilde{b}$.

H-differentiability of fuzzy-valued function due to M.L. Puri and D.A. Ralescu [11] is as follows

Definition 8 Let X be a subset of $\mathbb{R}$. A fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is said to be H-differentiable at $x^0 \in X$ if there exists a fuzzy number $D\tilde{f}(x^0)$ such that the limits (with respect to metric d_F)

$$\lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x^0 + h) \ominus_H \tilde{f}(x^0)], \quad \textit{and} \quad \lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x^0) \ominus_H \tilde{f}(x^0 - h)]$$

both exist and are equal to $D\tilde{f}(x^0)$. In this case, $D\tilde{f}(x^0)$ is called the H-derivative of $\tilde{f}$ at x^0. If $\tilde{f}$ is H-differentiable at any $x \in X$, we call $\tilde{f}$ is H-differentiable over X.

Remark 1 Many fuzzy-valued functions are H-differentaible for which Hukuhara differences $\tilde{f}(x^0+h) \ominus_H \tilde{f}(x^0)$ and $\tilde{f}(x^0) \ominus_H \tilde{f}(x^0-h)$ both exist. The following example illustrates the fact.

Example 1 Given in [11], let $\tilde{f} : (0, 2\pi) \to F(\mathbb{R})$ be defined on level sets by

$$[\tilde{f}(x)]_\alpha = (1-\alpha)(2+\sin(x))[-1,1],$$

for $\alpha \in [0,1]$. At $x^0 = \pi/2$, H-difference does not exist. Therefore, function is not H-differentiable at $x^0 = \pi/2$.

Now we prove following proposition regarding differentiability of $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$.

Proposition 3 *Let X be a subset of $\mathbb{R}$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at x^0 with derivative $D\tilde{f}(x^0)$, then $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ are differentiable at x^0, for all $\alpha \in [0,1]$. Moreover, we have $(D\tilde{f})_\alpha(x^0) = [D(\tilde{f}^L_\alpha)(x^0), D(\tilde{f}^U_\alpha)(x^0)]$.*

Proof The result follows from definitions of H-differentiability of fuzzy-valued function and metric on fuzzy number space. □

3.3 H-differentiability of fuzzy-valued function on $\mathbb{R}^n$

Definition 9 [10] Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset X of $\mathbb{R}^n$ and let $x^0 = (x_1^0, ..., x_n^0) \in X$ be fixed.
We say that $\tilde{f}$ has the i^{th} partial H-derivative $D_i\tilde{f}(x^0)$ at x^0 if the fuzzy-valued function $\tilde{g}(x_i) = \tilde{f}(x_1^0, .., x_{i-1}^0, x_i, x_{i+1}^0, .., x_n^0)$ is H-differentiable at x_i^0 with H-derivative $D_i\tilde{f}(x^0)$. We also write $D_i\tilde{f}(x^0)$ as $(\partial\tilde{f}/\partial x_i)(x^0)$.

Definition 10 [10] We say that $\tilde{f}$ is H-differentiable at x^0 if one of the partial H-derivatives $\partial\tilde{f}/\partial x_1, ..., \partial\tilde{f}/\partial x_n$ exists at x^0 and the remaining n-1 partial H-derivatives exist on some neighborhoods of x^0 and are continuous at x^0 (in the sense of fuzzy-valued function).

The gradient of $\tilde{f}$ at x^0 is denoted by

$$\nabla\tilde{f}(x^0) = (D_1\tilde{f}(x^0), ..., D_n\tilde{f}(x^0)),$$

and it defines a fuzzy-valued function from X to $F^n(\mathbb{R}) = F(\mathbb{R}) \times .... \times F(\mathbb{R})$ (n times), where each $D_i\tilde{f}(x^0)$ is a fuzzy number for i = 1,...,n. The α-level set of $\nabla\tilde{f}(x^0)$ is defined and denoted by

$$(\nabla\tilde{f}(x^0))_\alpha = ((D_1\tilde{f}(x^0))_\alpha, ..., (D_n\tilde{f}(x^0))_\alpha),$$

where

$$(D_i\tilde{f}(x^0))_\alpha = [D_i\tilde{f}_\alpha^L(x^0), D_i\tilde{f}_\alpha^U(x^0)],$$

i = 1,...,n.

We say that $\tilde{f}$ is H-differentiable on X if it is H-differentiable at every $x^0 \in X$.

Proposition 4 *Let X be an open subset of $\mathbb{R}^n$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable on X. Then $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are also differentiable on X, for all $\alpha \in [0,1]$. Moreover, for each $x \in X$, $(D_i\tilde{f}(x))_\alpha = [D_i\tilde{f}_\alpha^L(x), D_i\tilde{f}_\alpha^U(x)]$, i = 1,...,n.*

Proof The result follows from Propositions 2 and 3. □

Definition 11 We say that $\tilde{f}$ is continuously H-differentiable at x^0 if all of the partial H-derivatives $\partial\tilde{f}/\partial x_i$, i = 1,..,n, exist on some neighborhoods of x^0 and are continuous at x^0 (in the sense of fuzzy-valued function).
We say that $\tilde{f}$ is continuously H-differentiable on X if it is continuously H-differentiable at every $x^0 \in X$.

Proposition 5 *Let $\tilde{f} : X \to F(\mathbb{R})$ is continuously H-differentiable on X. Then $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are also continuously differentiable on X, for all $\alpha \in [0,1]$.*

Proof Followed by Propositions 2 and 4. □

Now we define twice continuously H-differentiable fuzzy-valued function.

Definition 12 Let $\tilde{f} : X \to F(\mathbb{R}), X \subset \mathbb{R}^n$ be a fuzzy-valued function. Suppose now that there is $x^0 \in X$ such that gradient of $\tilde{f}$, $\nabla \tilde{f}$, is itself H-differentiable at x^0, that is, for each i, the function $D_i \tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at x^0. Denote the H-partial derivative of $D_i \tilde{f}$ in the direction of $\bar{e}_j$ at x^0 by

$$D^2_{ij} \tilde{f} \quad or \quad \frac{\partial^2 \tilde{f}(x^0)}{\partial x_i \partial x_j}, \quad if \quad i \neq j,$$

and

$$D^2_{ii} \tilde{f} \quad or \quad \frac{\partial^2 \tilde{f}(x^0)}{\partial x_i^2}, \quad if \quad i = j.$$

Then we say that $\tilde{f}$ is twice H-differentiable at x^0, with second H-derivative $\nabla^2 \tilde{f}(x^0)$ which is denoted by

$$\nabla^2 \tilde{f}(x^0) = \begin{pmatrix} \frac{\partial^2 \tilde{f}(x^0)}{\partial x_1^2} & \dots & \frac{\partial^2 \tilde{f}(x^0)}{\partial x_1 \partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial^2 \tilde{f}(x^0)}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 \tilde{f}(x^0)}{\partial x_n^2} \end{pmatrix}$$

where $\frac{\partial^2 \tilde{f}(x^0)}{\partial x_i \partial x_j} \in F(\mathbb{R})$, i,j = 1,...,n.
If $\tilde{f}$ is twice H-differentiable at each x^0 in X, we say that $\tilde{f}$ is twice H-differentiable on X, and if for each i, j = 1,...,n, the cross-partial derivative $\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}$ is continuous function from X to $F(\mathbb{R})$, we say that $\tilde{f}$ is twice continuously H-differentiable on X.
The α-level set of $\nabla^2 \tilde{f}(x^0)$ is defined and denoted in matrix notation as

$$(\nabla^2 \tilde{f}(x^0))_\alpha = ((D^2_{ij} \tilde{f}(x^0))_\alpha)$$

i,j = 1,...,n and $\alpha \in [0,1]$, where $(D^2_{ij} \tilde{f}(x^0))_\alpha$ denotes α-cut of $(D^2_{ij} \tilde{f}(x^0))$.

Proposition 6 *Let $\tilde{f} : X \subseteq \mathbb{R}^n \to F(\mathbb{R})$ is differentiable with derivative $\nabla \tilde{f}$ on X and let each $D_i \tilde{f} : X \to F(\mathbb{R})$, i = 1,...,n, is also differentiable at x^0 with derivative $D^2_{ij} \tilde{f}(x^0)$, i , j = 1,...,n. Then $D_i \tilde{f}^L_\alpha$ and $D_i \tilde{f}^U_\alpha$ are also differentiable at x^0, for all $\alpha \in [0,1]$. Also, we have $(D^2_{ij} \tilde{f}(x^0))_\alpha = [D^2_{ij}(\tilde{f}^L_\alpha)(x^0), D^2_{ij}(\tilde{f}^U_\alpha)(x^0)]$, i,j = 1,...,n.*

Proof Follows by Proposition 5.

In order to define the Kuhn-Tucker like optimality conditions for nonlinear fuzzy optimization problems, we need to provide some properties of fuzzy-valued functions. For that first we state here following two Propositions from Real Analysis.

Proposition 7 *[13] Let ϕ be a real-valued function of two variables defined on $I \times [a,b]$, where I is an interval in $\mathbb{R}$. Suppose that the following conditions are satisfied:*

(i) For every $x \in I$, the real-valued function $h(y) = \phi(x,y)$ is Riemann integrable on [a,b]. In this case, we write $f(x) = \int_a^b \phi(x,y)\,dy$;
(ii) Let $x^0 \in int(I)$, the interior of I. For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{\partial \phi}{\partial x}(x,y) - \frac{\partial \phi}{\partial x}(x^0,y) \right| < \epsilon$$

for all $y \in [a,b]$ and all $x \in (x^0 - \delta, x^0 + \delta)$.

Then $\frac{\partial \phi}{\partial x}(x^0, y)$ *is Riemann integrable on [a,b],* $f'(x^0)$ *exists, and*

$$f'(x^0) = \int_a^b \frac{\partial \phi}{\partial x}(x^0, y)\, dy.$$

Proposition 8 *[2] Every function monotonic on an interval is Riemann integrable there.*

Let $\tilde{f} : X \to F(\mathbb{R})$ be a fuzzy-valued function defined on X subset of $\mathbb{R}^n$. Then for each $\alpha \in [0,1]$, $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are real-valued functions defined on X. For any fixed $x^0 \in X$, we have the corresponding real-valued functions $\tilde{f}^L_\alpha(x^0)$ and $\tilde{f}^U_\alpha(x^0)$ defined on $\alpha \in$[0,1]. By Proposition 1 and 8, we can easily say that $\tilde{f}^L_\alpha(x^0)$ and $\tilde{f}^U_\alpha(x^0)$ are riemann integrable. So, we define new functions F^L and F^U as follows

$$F^L(x) = \int_0^1 \tilde{f}^L_\alpha(x) d\alpha \;\; and \;\; F^U(x) = \int_0^1 \tilde{f}^U_\alpha(x) d\alpha \tag{3.1}$$

for every $x \in X$. Then we have following useful proposition.

Proposition 9 *[10] Let* $\tilde{f}$ *be a fuzzy-valued function defined on an open subset X of* $\mathbb{R}^n$*. If* $\tilde{f}$ *is continuously H-differentiable on some neighborhood of* x^0*. Then the real-valued functions* F^L *and* F^U *defined as above are continuously differentiable at* x^0 *and*

$$\frac{\partial F^L}{\partial x_i}(x^0) = \int_0^1 \frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(x^0)\, d\alpha \;\; and \;\; \frac{\partial F^U}{\partial x_i}(x^0) = \int_0^1 \frac{\partial \tilde{f}^U_\alpha}{\partial x_i}(x^0)\, d\alpha$$

for all i=1,..,n.

Proof We need to show that the partial derivatives $\frac{\partial F^L}{\partial x_i}$ and $\frac{\partial F^U}{\partial x_i}$ exist on some neighborhood of x^0 and are continuous at x^0 for all i =1,..,n. Since $\tilde{f}$ is continuously H-differentiable on some neighborhood of x^0. By Proposition 5, $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also continuously differentiable real-valued functions at x^0 for all $\alpha \in [0,1]$. Therefore, Proposition 8 say that

$$\frac{\partial F^L}{\partial x_i}(x^0) = \int_0^1 \frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(x^0)\, d\alpha \;\; and \;\; \frac{\partial F^U}{\partial x_i}(x^0) = \int_0^1 \frac{\partial \tilde{f}^U_\alpha}{\partial x_i}(x^0)\, d\alpha \tag{3.2}$$

for all i=1,..,n. Since $\frac{\partial f^L}{\partial x_i}$ is continuous at x^0, that is, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$||x - x^0|| < \delta \;\; implies \;\; \left| \frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(x) - \frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(x^0) \right| < \epsilon \;\; for\ all\ \alpha \in [0,1]$$

From (3.2), we have, if $||x - x^0|| < \delta$ then

$$\begin{aligned} \left| \frac{\partial F^L}{\partial x_i}(x^0) - \frac{\partial F^L}{\partial x_i}(x) \right| &= \left| \int_0^1 [\frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(x^0) - \frac{\partial \tilde{f}^U_\alpha}{\partial x_i}(x)]\, d\alpha \right| \\ &\leq \int_0^1 \left| \frac{\partial \tilde{f}^L_\alpha}{\partial x_i}(x^0) - \frac{\partial \tilde{f}^U_\alpha}{\partial x_i}(x) \right| d\alpha < \epsilon, \end{aligned}$$

for all i =1,...,n. Therefore, $\frac{\partial F^L}{\partial x_i}$ is contiuous for all i = 1,...,n. Similarly we can discuss the case of $\frac{\partial F^U}{\partial x_i}$. Hence complete the proof. □

4 Neccesary and sufficient optimality conditions for unconstrained fuzzy optimization problem

4.1 Unconstrained Fuzzy Optimization Problem

Let $T \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $\tilde{f}$ be fuzzy-valued function defined on T. Consider the following nonlinear fuzzy optimization problem

$$(FOP1) \qquad Minimize \;\; \tilde{f}(x) = \tilde{f}(x_1, .., x_n)$$
$$Subject \;\; to \;\; x \in T$$

We define here nondominated solutions of (FOP1).

Definition 13 Let T is an open subset of $\mathbb{R}^n$.

(i) A point $x^0 \in T$ is a locally nondominated solution of (FOP1) if there exists no $x^1(\neq x^0) \in N_\epsilon(x^0) \cap T$ such that $\tilde{f}(x^1) \prec \tilde{f}(x^0)$, $N_\epsilon(x^0)$ is a ϵ-neighborhood of x^0.
(ii) A point $x^0 \in T$ is a nondominated solution of (FOP1) if there exists no $x^1(\neq x^0) \in T$ such that $\tilde{f}(x^1) \prec \tilde{f}(x^0)$.
(iii) A point $x^0 \in T$ is a locally weak nondominated solution of (FOP1) if there exists no $x^1(\neq x^0) \in N_\epsilon(x^0) \cap T$ such that $\tilde{f}(x^1) \preceq \tilde{f}(x^0)$.
(iv) A point $x^0 \in T$ is a weak nondominated solution of (FOP1) if there exists no $x^1(\neq x^0) \in T$ such that $\tilde{f}(x^1) \preceq \tilde{f}(x^0)$.

4.2 Neccessary and sufficient optimality conditions

The first and second order neccessary and sufficient optimality conditions for real unconstrained optimization problem, given in [5], are as follows.

Theorem 1 *Let T is an open subset of $\mathbb{R}^n$.*

(i) ***(FONC)*** *Let f continuously differentiable function on T. If x^* is a local minimizer of f over T, then $\nabla f(x^*) = 0$.*
(ii) ***(SONC)*** *Let f twice continuously differentiable function on T. If x^* is a local minimizer of f over T, then $\nabla^2 f(x^*)$ is positive semidefinite.*
(iii) ***(SOSC)*** *Let f twice continuously differentiable function on T. Suppose that*
  1. $\nabla f(x^*) = 0$ and
  2. $\nabla^2 f(x^*)$ is positive definite.
  Then x^* is a strict local minimizer of f.

We prove here necessary and sufficient optimality conditions for obtaining nondominated solution of (FOP1). For that first we prove the following proposition.

Proposition 10 *If x^0 is a locally nondominated solution of (FOP1), then x^0 is also a local minimum of real-valued functions $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ for all $\alpha \in [0, 1]$.*

Proof We prove this result by contradiction. Assume that x^0 is not a local minimum of $\tilde{f}^L_\alpha$ or $\tilde{f}^U_\alpha$ for at least one $\alpha \in [0, 1]$. Without loss of generality, suppose that x^0 is not a local minimum of $\tilde{f}^L_\alpha$ for $\alpha_0 \in [0, 1]$. Therefore, there exists $x^1 \in N_\epsilon(x^0) \cap T$ such that

$$\tilde{f}^L_{\alpha_0}(x^1) < \tilde{f}^L_{\alpha_0}(x^0) \tag{4.1}$$

Since x^0 is a locally nondominated solution of (FOP1), there exists no $\bar{x} \in N_\epsilon(x^0) \cap T$ such that

$$\begin{cases} \tilde{f}^L_\alpha(\bar{x}) < \tilde{f}^L_\alpha(x^0) \\ \tilde{f}^U_\alpha(\bar{x}) \leq \tilde{f}^U_\alpha(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}) \leq \tilde{f}^L_\alpha(x^0) \\ \tilde{f}^U_\alpha(\bar{x}) < \tilde{f}^U_\alpha(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(\bar{x}) < \tilde{f}^L_\alpha(x^0) \\ \tilde{f}^U_\alpha(\bar{x}) < \tilde{f}^U_\alpha(x^0) \end{cases}$$

for all $\alpha \in [0,1]$. This gives contradiction to inequality (4.1). Therefore x^0 is also a minimum of real-valued functions $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ for all $\alpha \in [0,1]$. □

Now we prove the first-order necessary condition.

Theorem 2 *Suppose $x^0 \in T$ is a locally nondominated solution of (FOP1). Suppose also that $\tilde{f}$ is conituously H-differentiable function on T. Then*

$$\int_0^1 \nabla \tilde{f}^L_\alpha(x^0)d\alpha + \int_0^1 \nabla \tilde{f}^U_\alpha(x^0)d\alpha = 0$$

Proof The theorem can prove easily using Proposition 10 and Theorem 1 (i). □

Next, we prove first-order sufficient condition.

Theorem 3 *Let $\tilde{f}$ is twice-contiuously H-differentiable fuzzy-valued function defined on $T \subseteq \mathbb{R}^n$. If x^0 is a locally nondominated solution of (FOP1) then $\nabla^2 F(x^0)$ is positive semidefinite matrix.*

Here

$$\nabla^2 F(x^0) = \int_0^1 \nabla^2 \tilde{f}^L_\alpha(x^0)d\alpha + \int_0^1 \nabla^2 \tilde{f}^U_\alpha(x^0)d\alpha$$

Proof Since $\tilde{f}$ twice-contiuously H-differentiable fuzzy-valued function on T. By Proposition 2 and 6, $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also twice-contiuously H-differentiable functions on $\mathbb{R}^n$. Also, by Proposition 10, we say that $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ has local minimum at x^0. Therefore by second order neccessary condition for real unconstrained optimization stated in Theorem 1 (ii), $\nabla^2 \tilde{f}^L_\alpha(x^0)$ and $\nabla^2 \tilde{f}^U_\alpha(x^0)$ are positive semidefinite, for $\alpha \in [0,1]$.

That is, $x^T \cdot \nabla^2 \tilde{f}^L_\alpha(x^0) \cdot x \geq 0$ and $x^T \cdot \nabla^2 \tilde{f}^U_\alpha(x^0) \cdot x \geq 0$ for all $x \in T$, $x \neq 0$ and $\alpha \in [0,1]$, where x^T is transpose of x.

Therefore,

$$x^T \cdot \int_0^1 \nabla^2 \tilde{f}^L_\alpha(x^0) \cdot x \geq 0$$

and

$$x^T \cdot \int_0^1 \nabla^2 \tilde{f}^U_\alpha(x^0) \cdot x \geq 0$$

Adding these inequalities, we get

$$x^T \cdot \nabla^2 F(x^0) \cdot x \geq 0$$

for all $x \in T$, $x \neq 0$. Therefore, $\nabla^2 F(x^0)$ is positive semidefinite. □

Now, we prove second-order sufficient condition.

Theorem 4 *Let $\tilde{f}$ is twice contiuously H-differentiable function on $T \subseteq \mathbb{R}^n$. Suppose that*

1. $\nabla F(x^0)$
2. $\nabla^2 F(x^0)$ *is positive definite.*

Then, x^0 is locally weak nondominated solution of (FOP1).

Proof We prove this result using contradiction. Suppose $x^0 \in T$ is not a locally weak nondominated solution of (FOP1). Then, there exists $x^1 \in N_\epsilon(x^0) \cap T$ such that $\tilde{f}(x^1) \prec \tilde{f}(x^0)$. That is, there exists $x^1 \in N_\epsilon(x^0) \cap T$ such that

$$\begin{cases} \tilde{f}^L_\alpha(x^1) < \tilde{f}^L_\alpha(x^0) \\ \tilde{f}^U_\alpha(x^1) \leq \tilde{f}^U_\alpha(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(x^1) \leq \tilde{f}^L_\alpha(x^0) \\ \tilde{f}^U_\alpha(x^1) < \tilde{f}^U_\alpha(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}^L_\alpha(x^1) < \tilde{f}^L_\alpha(x^0) \\ \tilde{f}^U_\alpha(x^1) < \tilde{f}^U_\alpha(x^0) \end{cases}$$

for all $\alpha \in [0, 1]$. Therefore, we have

$$F(x^1) < F(x^0), \tag{4.2}$$

where

$$F(x) = \int_0^1 \tilde{f}^L_\alpha(x)d\alpha + \int_0^1 \tilde{f}^U_\alpha(x)d\alpha.$$

As $\tilde{f}$ is twice continuously H-differentaible function, F(x) is also. Using assumption 2 and Rayleigh's inequality (refer [5]), it follows that if $d \neq 0$, then

$$0 < \lambda_{\min}(F(x^0)\|d\|^2 \leq d^T \cdot F(x^0) \cdot d.$$

By Taylor's theorem and assumption 1,

$$\begin{aligned} F(x^0 + d) - F(x^0) &= \frac{1}{2} d^T \cdot \nabla^2 F(x^0) \cdot d + O(\|d\|^2) \\ &\geq \frac{\lambda_{\min}(\nabla^2 F(x^0))}{2}\|d\|^2 + O(\|d\|^2). \end{aligned}$$

Hence, for all d such that $\|d\|$ is sufficiently small,

$$F(x^0 + d) > F(x^0)$$

This gives contradiction to inequality (4.2). Hence proved the result. □

We give one example to illustrate the above results.

Example 2 Let $\tilde{f} : \mathbb{R}^2 \to F(\mathbb{R})$ be defined by $\tilde{f}(x_1, x_2) = (0, 2, 4) \odot x_1^2 \oplus (0, 2, 4) \odot x_2^2 \odot (1, 3, 5)$, where $(0, 2, 4)$ and $(1, 3, 5)$ are triangular fuzzy numbers.

By first order neccessary condition: $\nabla F(x) = 0$.

Here, $\tilde{f}^L_\alpha(x_1, x_2) = 2\alpha x_1^2 + 2\alpha x_2^2 + (1 + 2\alpha)$ and

$\tilde{f}^U_\alpha(x_1, x_2) = (4 - 2\alpha)x_1^2 + (4 - 2\alpha)x_2^2 + (5 - 2\alpha)$. Therefore,

$$\nabla f^L_\alpha(x_1, x_2) = \begin{pmatrix} 4x_1\alpha \\ 4x_2\alpha \end{pmatrix}$$

and

$$\nabla f_{\alpha}^{U}(x_1, x_2) = \begin{pmatrix} 2(4-2\alpha)x_1 \\ 2(4-2\alpha)x_2 \end{pmatrix}$$

Therefore,

$$\nabla F(x_1, x_2) = \begin{pmatrix} 14x_1 \\ 14x_2 \end{pmatrix}$$

This implies , $x^0 = (x_1, x_2) = (0, 0)$.
Now to verify second order necessary and sufficient conditions, we find $\nabla^2 F(x)$:

$$\nabla^2 \tilde{f}_{\alpha}^{L}(x) = \begin{pmatrix} 4\alpha & 0 \\ 0 & 4\alpha \end{pmatrix}$$

and

$$\nabla^2 \tilde{f}_{\alpha}^{U}(x) = \begin{pmatrix} 2(4-2\alpha) & 0 \\ 0 & 2(4-2\alpha) \end{pmatrix}$$

Therefore,

$$\nabla^2 F(x) = \begin{pmatrix} 14 & 0 \\ 0 & 14 \end{pmatrix}$$

which is positive definite. Therefore, by neccessary and sufficient conditions, $x^0 = (0, 0)$ is a nondominated solution of given problem.

5 Neccesary and sufficient optimality conditions for constrained fuzzy optimization problem

5.1 Constrained Fuzzy Optimization Problem

Let $T \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $\tilde{f}, \tilde{g}_j$, for j = 1,...,m be fuzzy-valued functions defined on T. Consider the following nonlinear fuzzy optimization problem

$$(FOP2) \qquad Minimize \;\; \tilde{f}(x) = \tilde{f}(x_1, .., x_n)$$

$$Subject \;\; to \;\; \tilde{g}_j(x) \preceq \tilde{0}, \quad j = 1, .., m,$$

where $\tilde{0}$ is a fuzzy number defined as $\tilde{0}(r) = 1$ if $r = 0$ and $\tilde{0}(r) = 0$ if $r \neq 0$ and its level set is $\tilde{0}_{\alpha} = \{0\}$ for $\alpha \in [0, 1]$.

Definition 14 Let $x^0 \in X = \{x \in T : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, .., m\}$. We say that x^0 is a nondominated solution of (FOP2) if there exists no $x^1 (\neq x^0) \in X$ such that $\tilde{f}(x^1) \prec \tilde{f}(x^0)$. That is, x^0 is a nondominated solution of (FOP2) if there exists no $x^1 (\neq x^0) \in X$ such that

$$\begin{cases} \tilde{f}_{\alpha}^{L}(x^1) < \tilde{f}_{\alpha}^{L}(x^0) \\ \tilde{f}_{\alpha}^{U}(x^1) \leq \tilde{f}_{\alpha}^{U}(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}_{\alpha}^{L}(x^1) \leq \tilde{f}_{\alpha}^{L}(x^0) \\ \tilde{f}_{\alpha}^{U}(x^1) < \tilde{f}_{\alpha}^{U}(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}_{\alpha}^{L}(x^1) < \tilde{f}_{\alpha}^{L}(x^0) \\ \tilde{f}_{\alpha}^{U}(x^1) < \tilde{f}_{\alpha}^{U}(x^0) \end{cases}$$

for all $\alpha \in [0, 1]$.

5.2 Necessary and Sufficient Optimality Conditions

Let f and $g_j, j = 1, .., m$, be real-valued functions defined on $T \subset \mathbb{R}^n$. Then we consider the following optimization problem

$$(P) \qquad Minimize \;\; f(x) = f(x_1, .., x_n)$$
$$Subject \;\; to \;\; g_j(x) \leq 0, \quad j = 1, .., m.$$

The well-known Kuhn-Tucker optimality conditions for problem (P) by S. Rangarajan in [12] is stated as follows:

Theorem 5 *Let f be a convex, continuously differentiable function mapping T into* $\mathbb{R}$*, where* $T \subset \mathbb{R}^n$ *is open and convex. For j =1,..m, the constraint functions* $g_j : T \to \mathbb{R}$ *are convex, continuously differentiable functions. Suppose there is some* $x \in T$ *such that* $g_j(x) < 0$*, j =1,..,m.*
Then x^0 *is an optimal solution of propositionblem (P) over the feasible set* $\{x \in T : g_j(x) \leq 0, j = 1, .., m\}$ *if and only if there exist multipliers* $0 \leq \mu_j \in \mathbb{R}$*, j = 1,..,m, such that the Kuhn-Tucker first order conditions hold:*

(KT-1) $\nabla f(x^0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^0) = 0$;
(KT-2) $\mu_j \cdot g_j(x^0) = 0$ *for all j = 1,..,m.*

First we introduce the concept of convexity for fuzzy-valued functions.

Definition 15 Let T be a convex subset of $\mathbb{R}^n$ and $\tilde{f}$ be a fuzzy-valued function defined on T. We say that $\tilde{f}$ is **convex** at x^0 if

$$\tilde{f}(\lambda x^0 + (1-\lambda)x) \preceq (\lambda \odot \tilde{f}(x^0) \oplus ((1-\lambda) \odot \tilde{f}(x))$$

for each $\lambda \in (0, 1)$ and $x \in T$.

Proposition 11 $\tilde{f} : T \to F(\mathbb{R})$ *is convex at* x^0 *if and only if* $\tilde{f}^L_\alpha$ *and* $\tilde{f}^U_\alpha$ *are convex at* x^0*, for all* $\alpha \in [0, 1]$.

Proof The result can prove easily using the concepts of arithmetic operations and partial order relation of fuzzy numbers. □

We now present the Kuhn-Tucker like optimality conditions for (FOP2).

Theorem 6 *Let the fuzzy-valued objective function* $\tilde{f} : T \to F(\mathbb{R})$ *is convex and continuously H-differentiable, where* $T \subset \mathbb{R}^n$ *is open and convex. For j = 1,..,m, the fuzzy-valued constraint functions* $\tilde{g}_j : T \to F(\mathbb{R})$ *are convex and continuously H-differentiable. Let* $X = \{x \in T \subset \mathbb{R}^n : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, .., m\}$ *be a feasible set of problem (FOP) and let* $x^0 \in X$*. Suppose there is some* $x \in T$ *such that* $\tilde{g}_j(x) \prec \tilde{0}$*, j =1,..,m.*
Then x^0 *is a nondominated solution of problem (FOP2) over X if and only if there exist multipliers* $0 \leq \mu_j \in \mathbb{R}$*, j = 1,..,m, such that the Kuhn-Tucker first order conditions hold:*

(FKT-1) $\displaystyle\int_0^1 \nabla \tilde{f}^L_\alpha(x^0)\, d\alpha + \int_0^1 \nabla \tilde{f}^U_\alpha(x^0)\, d\alpha + \sum_{j=1}^{m} \mu_j \nabla \tilde{g}^U_{j0}(x^0) = 0$;
(FKT-2) $\mu_j \cdot \tilde{g}^U_{j0}(x^0) = 0$ *for all j = 1,..,m.*

Proof **Necessary.** Define a new function,

$$F(x) = \int_0^1 \tilde{f}_\alpha^L(x)d\alpha + \int_0^1 \tilde{f}_\alpha^U(x)d\alpha. \tag{5.1}$$

Since $\tilde{f}$ is convex and continuously H-differentiable function, by Propositions 5 and 11, we say that F(x) is convex and continuously differentiable real-valued function on T. Since x^0 is a nondominated solution of (FOP2). Then there exists no $(x^1 \neq x^0) \in X$ such that

$$\begin{cases} \tilde{f}_\alpha^L(x^1) < \tilde{f}_\alpha^L(x^0) \\ \tilde{f}_\alpha^U(x^0) \le \tilde{f}_\alpha^U(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}_\alpha^L(x^1) \le \tilde{f}_\alpha^L(x^0) \\ \tilde{f}_\alpha^U(x^1) < \tilde{f}_\alpha^U(x^0) \end{cases} \text{or} \begin{cases} \tilde{f}_\alpha^L(x^1) < \tilde{f}_\alpha^L(x^0) \\ \tilde{f}_\alpha^U(x^1) < \tilde{f}_\alpha^U(x^0) \end{cases}$$

for all $\alpha \in [0,1]$.

That is, there exists no $x^1(\neq x^0) \in X$ such that

$$F(x^1) < F(x^0)$$

Therefore,

$$F(x^0) \le F(x^1)$$

Since $\tilde{g}_j$ are convex and continuously H-differentiable functions for j = 1,..,m implies $\tilde{g}_{j\alpha}^L$ and $\tilde{g}_{j\alpha}^U$ are real-valued convex and continuously differentiable functions for all $\alpha \in [0,1]$ and j =1,..,m.

By definition of partial ordering and Proposition 1, we have

$$\begin{aligned} X &= \{x \in T \subset \mathbb{R}^n : \tilde{g}_j(x) \preceq \tilde{0}, j = 1, ..., m\} \\ &= \{x \in T \subset \mathbb{R}^n : \tilde{g}_{j\alpha}^L(x) \le 0 \; and \; \tilde{g}_{j\alpha}^U(x) \le 0, j = 1, ..., m\} \\ &= \{x \in T \subset \mathbb{R}^n : \tilde{g}_{j\alpha}^U(x) \le 0, j = 1, ..., m\} \\ &= \{x \in T \subset \mathbb{R}^n : \tilde{g}_{j0}^U(x) \le 0, j = 1, ..., m\} \end{aligned}$$

Therefore, $x^0 \in X = \{x \in T \subset \mathbb{R}^n : \tilde{g}_{j0}^U(x) \le 0, j = 1, .., m\}$ and there is some $x \in T$ such that $\tilde{g}_{j0}^U(x) < 0$, j =1,..,m. So our problem becomes an optimization problem with real objective function F(x) subject to real constraints.

Therefore, by Theorem 5, there exist multipliers $0 \le \mu_j \in \mathbb{R}$, j = 1,..,m, such that the following kuhn-Tucker first order conditions hold:

(KT-1) $\nabla F(x^0) + \sum_{j=1}^m \mu_j \nabla \tilde{g}_{j0}^U(x^0) = 0$;

(KT-2) $\mu_j \cdot \tilde{g}_{j0}^U(x^0) = 0$ for all j = 1,..,m.

But $F(x) = \int_0^1 \tilde{f}_\alpha^L(x)d\alpha + \int_0^1 \tilde{f}_\alpha^U(x)d\alpha$. We obtain the kuhn-Tucker conditions for problem (FOP2) as follows

(FKT-1) $\int_0^1 \nabla \tilde{f}_\alpha^L(x^0)\, d\alpha + \int_0^1 \nabla \tilde{f}_\alpha^U(x^0)\, d\alpha + \sum_{j=1}^{m} \mu_j \nabla \tilde{g}_{j0}^U(x^0) = 0$;

(FKT-2) $\mu_j \cdot \tilde{g}_{j0}^U(x^0) = 0$ for all j = 1,..,m.

Sufficient. We are going to prove this part by contradiction. Suppose that x^0 not a nondominated solution. Then there exists a $x^1(\neq x^0) \in X$ such that $\tilde{f}(x^1) \prec \tilde{f}(x^0)$. Therefore, we have

$$\tilde{f}_\alpha^L(x^1) + \tilde{f}_\alpha^U(x^1) < \tilde{f}_\alpha^L(x^0) + \tilde{f}_\alpha^U(x^0)$$

for all $\alpha \in [0,1]$. From (5.1), we obtain

$$F(x^1) < F(x^0) \tag{5.2}$$

Since F is convex and continuously differentiable function. Furthermore, $x^0 \in X = \{x \in T \subset \mathbb{R}^n : \tilde{g}_{j0}^U(x) \leq 0, j = 1,..,m\}$, by conditions (FKT-1) and (FKT-2) of this theorem, we obtain the following new conditions:

(KT-1) $\nabla F(x^0) + \sum_{j=1}^m \mu_j \nabla \tilde{g}_{j0}^U(x^0) = 0$;
(KT-2) $\mu_j \cdot \tilde{g}_{j0}^U(x^0) = 0$ for all j = 1,..,m.

Using Theorem 5, we say that x^0 is an optimal solution of real-objective function F with real constraints $\tilde{g}_{j0}^U(x) \leq 0$, for $j = 1,..,m$. i.e., $F(x^0) \leq F(x^1)$, which contradicts to (5.2). Hence the proof. □

We consider here first fuzzy optimization problem having fuzzy-valued objective function and real constraints.

Example 3

$$\begin{aligned} Minimize \quad & \tilde{f}(x_1,x_2) = (\tilde{a} \odot x_1^2) \oplus (\tilde{b} \odot x_2^2) \\ subject\ to\ & g(x_1,x_2) = (x_1-2)^2 + (x_2-2)^2 \leq 1, \end{aligned}$$

where $\tilde{a} = (1,2,3)$ and $\tilde{b} = (0,1,2)$ are triangular fuzzy numbers defined on $\mathbb{R}$ as

$$\tilde{a}(r) = \begin{cases} (r-1), & if\ 1 \leq r \leq 2, \\ (3-r), & if\ 2 < r \leq 3, \\ 0 & otherwise \end{cases} \quad \tilde{b}(r) = \begin{cases} r, & if\ 0 \leq r \leq 1, \\ 2-r, & if\ 1 < r \leq 2, \\ 0 & otherwise \end{cases}$$

Using definition 2, we obtain

$\tilde{f}_\alpha^L(x_1,x_2) = (1+\alpha)x_1^2 + \alpha x_2^2$ and $\tilde{f}_\alpha^U(x_1,x_2) = (3-\alpha)x_1^2 + (2-\alpha)x_2^2$
for $\alpha \in [0,1]$.

We obtain

$$\nabla \tilde{f}_\alpha^L(x_1,x_2) = \begin{pmatrix} 2x_1(\alpha+1) \\ 2x_2\alpha \end{pmatrix}, \nabla \tilde{f}_\alpha^U(x_1,x_2) = \begin{pmatrix} 2x_1(3-\alpha) \\ 2x_2(2-\alpha) \end{pmatrix} \text{ and}$$
$$\nabla g(x_1,x_2) = \begin{pmatrix} 2(x_1-2) \\ 2(x_2-2) \end{pmatrix}$$

Therefore, we have

$$\int_0^1 \nabla \tilde{f}_\alpha^L(x_1,x_2)\, d\alpha = \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix}, \int_0^1 \nabla \tilde{f}_\alpha^U(x_1,x_2)\, d\alpha = \begin{pmatrix} 5x_1 \\ 3x_2 \end{pmatrix}.$$

From Theorem 6, we have the following Kuhn-Tucker conditions

(FKT-1) $8x_1 + 2\mu(x_1 - 2) = 0$, $4x_2 + 2\mu(x_2 - 2) = 0$,
(FKT-2) $\mu((x_1 - 2)^2 + (x_2 - 2)^2 - 1) = 0$.

Solving these equations, we get the solution $(x_1, x_2) = (6/5, 3/2)$ and $\mu = 6$. By Theorem 6, we say that $(x_1^*, x_2^*) = (6/5, 3/2)$ is nondominated solution for given problem . Also the minimum value of objective function is $\tilde{f}_{min} = (1.44, 5.13, 8.82)$ and we can find its defuzzified value 5.13 by using center of area method (ref. [7]).

Now we solve the same fuzzy optimization problem having fuzzy-valued objective function with fuzzy constraints.

Example 4

$$\begin{aligned} &Minimize \quad \tilde{f}(x_1, x_2) = (\tilde{a} \odot x_1^2) \oplus (\tilde{b} \odot x_2^2) \\ &subject\ to\ \tilde{g}(x_1, x_2) = (\tilde{b} \odot (x_1 - 2)^2) \oplus (\tilde{b} \odot (x_2 - 2)^2) \preceq \tilde{c}, \end{aligned}$$

where $\tilde{a} = (1, 2, 3)$, $\tilde{b} = (0, 1, 2)$ and $\tilde{c} = (0, 2, 4)$ are triangular fuzzy numbers defined on $\mathbb{R}$ as

$$\tilde{a}(r) = \begin{cases} (r-1), & if\ 1 \le r \le 2, \\ (3-r), & if\ 2 < r \le 3, \\ 0 & otherwise \end{cases} \quad \tilde{b}(r) = \begin{cases} r, & if\ 0 \le r \le 1, \\ 2-r, & if\ 1 < r \le 2, \\ 0 & otherwise \end{cases}$$

$$\tilde{c}(r) = \begin{cases} r/2, & if\ 0 \le r \le 2, \\ (4-r)/2, & if\ 2 < r \le 4, \\ 0 & otherwise \end{cases}$$

Using definition 2, we obtain

$\tilde{f}_\alpha^L(x_1, x_2) = (1 + \alpha)x_1^2 + \alpha x_2^2$ and $\tilde{f}_\alpha^U(x_1, x_2) = (3 - \alpha)x_1^2 + (2 - \alpha)x_2^2$ for $\alpha \in [0, 1]$.

Moreover, $\tilde{g}_\alpha^U(x_1, x_2) = (2 - \alpha)(x_1 - 2)^2 + (2 - \alpha)(x_2 - 2)^2 \le (4 - 2\alpha)$ for $\alpha \in [0, 1]$.

Therefore, $\tilde{g}_0^U(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 2)^2 \le 2$.
Now we obtain

$$\nabla \tilde{f}_\alpha^L(x_1, x_2) = \begin{pmatrix} 2x_1(\alpha + 1) \\ 2x_2\alpha \end{pmatrix}, \ \nabla \tilde{f}_\alpha^U(x_1, x_2) = \begin{pmatrix} 2x_1(3 - \alpha) \\ 2x_2(2 - \alpha) \end{pmatrix} \text{ and}$$

$$\nabla g(x_1, x_2) = \begin{pmatrix} 2(x_1 - 2) \\ 2(x_2 - 2) \end{pmatrix}$$

Therefore, we have

$$\int_0^1 \nabla \tilde{f}_\alpha^L(x_1, x_2)\, d\alpha = \begin{pmatrix} 3x_1 \\ x_2 \end{pmatrix}, \ \int_0^1 \nabla \tilde{f}_\alpha^U(x_1, x_2)\, d\alpha = \begin{pmatrix} 5x_1 \\ 3x_2 \end{pmatrix}.$$

From Theorem 6, we have the following Kuhn-Tucker conditions

(FKT-1) $8x_1 + 2\mu(x_1 - 2) = 0$, $4x_2 + 2\mu(x_2 - 2) = 0$,
(FKT-2) $\mu((x_1 - 2)^2 + (x_2 - 2)^2 - 2) = 0$.

Solving these equations, we get the solution $(x_1, x_2) = ((-6+2\sqrt{41})/(1+\sqrt{41}), (-6+2\sqrt{41})/(-1+\sqrt{41}))$ and $\mu = -3+\sqrt{41}$. By Theorem 6, we say that $(x_1^*, x_2^*) = ((-6+2\sqrt{41})/(1+\sqrt{41}), (-6+2\sqrt{41})/(-1+\sqrt{41}))$ is nondominated solution for given problem . Also the minimum value of objective function is $\tilde{f}_{min} = (0.8453, 3.2773, 5.7094)$ and we can find its defuzzified value 3.2773 by using center of area method.

Remark 2 By comparing the defuzzified value of minimum objective functions in example 3 and 4, we observe that there is significant effect on minimum value of the fuzzy-valued objective function if consider fuzzy optimization problem with fuzzy constraints. Moreover, if we consider the fuzzy optimization problem having fuzzy constraints then we can not apply Theorem 6.2 from [10] to find the nondominated solution. In that case, our result is quite useful to get the solution.

6 Conclusion

Using partial order relation on fuzzy number space, the necessary and sufficient Kuhn-Tucker like optimality condtions for nonlinear fuzzy optimization problem have been derived in this paper. We have used hukuhara differentiability and convexity of fuzzy-valued function for proving the same. We have also provided an example to illustrate the possible applications in this subject.

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