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NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR NONLINEAR UNCONSTRAINED FUZZY OPTIMIZATION PROBLEM

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Abstract. Nonlinear unconstrained fuzzy optimization problem is considered in this paper. Using the concept of convexity and Hukuhara differentiability of fuzzy-valued functions, the necessary and sufficient optimality conditions are derived.

1. Introduction

Crisp optimization techniques have been successfully applied for years. In real process optimization, there exist different types of uncertainties in the system. Zimmermann [17] pointed out various kinds of uncertainties that can be categorized as stochastic uncertainty or fuzziness. The optimization under a fuzzy environment or which involve fuzziness is called fuzzy optimization.

Bellman and Zadeh in 1970 [2] proposed the concept of fuzzy decision and the decision model under fuzzy environments. After that, various approaches to fuzzy linear and nonlinear optimization, have been developed over the years by researchers.

In this paper, we establish first and second order necessary and sufficient optimality conditions for obtaining the non-dominated solution of a nonlinear unconstrained fuzzy optimization problem.

In Section 2, we cite some of the basic definitions regarding fuzzy numbers. In Section 3, we consider the differential calculus of fuzzy-valued functions defined on $\mathbb{R}$ and $\mathbb{R}^n$ using Hukuhara differentiability of fuzzy-valued functions. In Section 4, we propose the first and second order optimality conditions to find a non-dominated solution of a unconstrained fuzzy optimization problem. At last we conclude in Section 5.

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2. Preliminaries

Definition 2.1. [7] *Let $\mathbb{R}$ be the set of real numbers and $\tilde{a} : \mathbb{R} \to [0,1]$ be a fuzzy set. We say that $\tilde{a}$ is a fuzzy number if it satisfies the following properties:*

(i) $\tilde{a}$ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $\tilde{a}(x_0) = 1$;
(ii) $\tilde{a}$ is fuzzy convex, that is, $\tilde{a}(tx + (1-t)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$, whenever $x, y \in \mathbb{R}$ and $t \in [0,1]$;
(iii) $\tilde{a}(x)$ is upper semi-continuous on $\mathbb{R}$, that is, $\{x/\tilde{a}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0,1]$;
(iv) $cl\{x \in \mathbb{R}/(a)(x) > 0\}$ forms a compact set, where, 'cl' denotes the closure.

The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. For all $\alpha \in (0,1]$, α-level set $\tilde{a}_\alpha$ of any $\tilde{a} \in F(\mathbb{R})$ is defined as $\tilde{a}_\alpha = \{x \in \mathbb{R}/\tilde{a}(x) \geq \alpha\}$.
The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$.

By the definition of fuzzy numbers, we can prove that, for any $\tilde{a} \in F(\mathbb{R})$ and for each $\alpha \in (0,1]$, $\tilde{a}_\alpha$ is compact convex subset of $\mathbb{R}$ and we write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. $\tilde{a} \in F(\mathbb{R})$ can be recovered from its α-level sets by a well-known decomposition theorem (ref. [8]), which states that $\tilde{a} = \cup_{\alpha \in [0,1]} \alpha \cdot \tilde{a}_\alpha$ where union on the right-hand side is the standard fuzzy union.

Definition 2.2. [13] *According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space $F(\mathbb{R})$ by their α-level sets as follows:*

$$(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha],$$
$$(\lambda \odot \tilde{a})_\alpha = [\lambda \cdot \tilde{a}^L_\alpha, \lambda \cdot \tilde{a}^U_\alpha],$$

where $\tilde{a}, \tilde{b} \in F(\mathbb{R})$, $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$.

Definition 2.3. [15] *Let $A, B \subseteq \mathbb{R}^n$. The Hausdorff metric d_H is defined by*

$$d_H(A,B) = \max\{\sup_{x \in A} \inf_{y \in B} ||x-y||, \sup_{y \in B} \inf_{x \in A} ||x-y||\}.$$

Then the metric d_F on $F(\mathbb{R})$ is defined as

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0 \leq \alpha \leq 1} \{d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)\},$$

for all $\tilde{a}, \tilde{b} \in F(\mathbb{R})$. Since $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are closed bounded intervals in $\mathbb{R}$,

$$d_F(\tilde{a}, \tilde{b}) = \sup_{0 \leq \alpha \leq 1} \max\{|\tilde{a}^L_\alpha - \tilde{b}^L_\alpha|, |\tilde{a}^U_\alpha - \tilde{b}^U_\alpha|\}.$$

We need the following proposition.

Proposition 2.1. [5] *For $\tilde{a} \in F(\mathbb{R})$, we have*

(i) $\tilde{a}_\alpha^L$ *is bounded left continuous nondecreasing function on (0,1];*
(ii) $\tilde{a}_\alpha^U$ *is bounded left continuous non-increasing function on (0,1];*
(iii) $\tilde{a}_\alpha^L$ *and* $\tilde{a}_\alpha^U$ *are right continuous at* $\alpha = 0$*;*
(iv) $\tilde{a}_\alpha^L \leq \tilde{a}_\alpha^U$.

Moreover, if the pair of functions $\tilde{a}_\alpha^L$ *and* $\tilde{a}_\alpha^U$ *satisfy the conditions (i)-(iv), then there exists a unique* $\tilde{a} \in F(\mathbb{R})$ *such that* $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$*, for each* $\alpha \in [0, 1]$.

We define here a partial order relation on the set of fuzzy numbers called fuzzy-max order, introduced by Ramík and Rimanek [10].

Definition 2.4. *For* $\tilde{a}$ *and* $\tilde{b}$ *in* $F(\mathbb{R})$ *and* $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ *and* $\tilde{b}_\alpha = [\tilde{b}_\alpha^L, \tilde{b}_\alpha^U]$ *are two closed intervals in* $\mathbb{R}$*, for all* $\alpha \in [0, 1]$*, we define*

(i) $\tilde{a} \preceq \tilde{b}$ *if and only if* $\tilde{a}_\alpha^L \leq \tilde{b}_\alpha^L$ *and* $\tilde{a}_\alpha^U \leq \tilde{b}_\alpha^U$ *for all* $\alpha \in [0, 1]$*;*
(ii) $\tilde{a} \prec \tilde{b}$ *if and only if* $\tilde{a} \preceq \tilde{b}$ *and there exists an* $\alpha_0 \in [0, 1]$ *such that* $\tilde{a}_{\alpha_0}^L < \tilde{b}_{\alpha_0}^L$ *or* $\tilde{a}_{\alpha_0}^U < \tilde{b}_{\alpha_0}^U$.

"$\preceq$*" is a partial order relation on the set of fuzzy numbers.*

Now we define comparable fuzzy numbers as follows.

Definition 2.5. *For* $\tilde{a}$, $\tilde{b}$ *in* $F(\mathbb{R})$*, we say that* $\tilde{a}$ *and* $\tilde{b}$ *are comparable if either* $\tilde{a} \preceq \tilde{b}$ *or* $\tilde{b} \preceq \tilde{a}$.

Definition 2.6. [12] *The membership function of a triangular fuzzy number* $\tilde{a}$ *is defined as*

$$\mu_{\tilde{a}}(r) = \begin{cases} \frac{(r-a^L)}{(a-a^L)}, & \text{if } a^L \leq r \leq a \\ \frac{(a^U-r)}{(a^U-a)}, & \text{if } a < r \leq a^U \\ 0, & \text{otherwise} \end{cases}$$

which is denoted by

$$\tilde{a} = (a^L, a, a^U).$$

The α*-level set of* $\tilde{a}$ *is then*

$$\tilde{a}_\alpha = [(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a].$$

3. Differential calculus of fuzzy-valued function

3.1. **Continuity of fuzzy-valued function.**

Definition 3.1. [14] *Let* V *be a real vector space and* $F(\mathbb{R})$ *be a fuzzy number space. Then a function* $\tilde{f} : V \to F(\mathbb{R})$ *is called a fuzzy-valued function defined on* V.

Corresponding to such a function $\tilde{f}$ and $\alpha \in [0,1]$, we define two real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ on V as $\tilde{f}^L_\alpha(x) = (\tilde{f}(x))^L_\alpha$ and $\tilde{f}^U_\alpha(x) = (\tilde{f}(x))^U_\alpha$ for all $x \in V$.

Definition 3.2. [6] *Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. We say that $\tilde{f}$ is continuous at $c \in \mathbb{R}^n$ if for every $\epsilon > 0$, there exists a $\delta = \delta(c, \epsilon) > 0$ such that*

$$d_F(\tilde{f}(x), \tilde{f}(c)) < \epsilon$$

for all $x \in \mathbb{R}^n$ with $\|x - c\| < \delta$. That is,

$$\lim_{x \to c} \tilde{f}(x) = \tilde{f}(c).$$

We prove the following proposition.

Proposition 3.1. *Let $\tilde{f} : \mathbb{R}^n \to F(\mathbb{R})$ be a fuzzy-valued function. If $\tilde{f}$ is continuous at $c \in \mathbb{R}^n$, then functions $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ are continuous at c, for all $\alpha \in [0,1]$.*

Proof: The result follows by using the definitions of continuity of fuzzy-valued function $\tilde{f}$ and metric on fuzzy numbers.

3.2. **H-differentiability of fuzzy-valued function on $\mathbb{R}$.**

Definition 3.3. *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If there exists a fuzzy number $\tilde{c}$ such that $\tilde{c} \oplus \tilde{b} = \tilde{a}$. Then $\tilde{c}$ is called Hukuhara difference of $\tilde{a}$ and $\tilde{b}$ and is denoted by $\tilde{a} \ominus_H \tilde{b}$.*

Hukuhara differentiability (H-differentiability) of a fuzzy-valued function due to Puri and Ralescu [9] is as follows.

Definition 3.4. *Let X be a subset of $\mathbb{R}$. A fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is said to be H-differentiable at $x^0 \in X$ if there exists a fuzzy number $D\tilde{f}(x^0)$ such that the limits (with respect to metric d_F)*

$$\lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x^0 + h) \ominus_H \tilde{f}(x^0)], \text{ and } \lim_{h \to 0^+} \frac{1}{h} \odot [\tilde{f}(x^0) \ominus_H \tilde{f}(x^0 - h)]$$

both exist and are equal to $D\tilde{f}(x^0)$. In this case, $D\tilde{f}(x^0)$ is called the H-derivative of $\tilde{f}$ at x^0. If $\tilde{f}$ is H-differentiable at any $x \in X$, we call $\tilde{f}$ is H-differentiable over X.

We prove following proposition regarding differentiability of $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$.

Proposition 3.2. *Let X be a subset of $\mathbb{R}$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at x^0 with H-derivative $D\tilde{f}(x^0)$, then $\tilde{f}^L_\alpha(x)$ and $\tilde{f}^U_\alpha(x)$ are differentiable at x^0, for all $\alpha \in [0,1]$. Moreover, we have $(D\tilde{f})_\alpha(x^0) = [D(\tilde{f}^L_\alpha)(x^0), D(\tilde{f}^U_\alpha)(x^0)]$.*

Proof: The result follows from the definitions of H-differentiability of fuzzy-valued function and metric on fuzzy number space.

3.3. **H-differentiability of fuzzy-valued function on $\mathbb{R}^n$.**

Definition 3.5. [16] *Let $\tilde{f}$ be a fuzzy-valued function defined on an open subset X of $\mathbb{R}^n$ and let $\bar{x}^0 = (x_1^0, \ldots, x_n^0) \in X$ be fixed. We say that $\tilde{f}$ has the i^{th} partial H-derivative $D_i\tilde{f}(\bar{x}^0)$ at $\bar{x}^0$ if the fuzzy-valued function $\tilde{g}(x_i) = \tilde{f}(x_1^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0)$ is H-differentiable at x_i^0 with H-derivative $D_i\tilde{f}(\bar{x}^0)$. We also write $D_i\tilde{f}(\bar{x}^0)$ as $(\partial\tilde{f}/\partial x_i)(\bar{x}^0)$.*

Definition 3.6. [16] *We say that $\tilde{f}$ is H-differentiable at $\bar{x}^0$ if one of the partial H-derivatives $\partial\tilde{f}/\partial x_1, \ldots, \partial\tilde{f}/\partial x_n$ exists at $\bar{x}^0$ and the remaining $n-1$ partial H-derivatives exist on some neighborhoods of $\bar{x}^0$ and are continuous at $\bar{x}^0$ (in the sense of fuzzy-valued function).*

The gradient of $\tilde{f}$ at $\bar{x}^0$ is denoted by

$$\nabla\tilde{f}(\bar{x}^0) = (D_1\tilde{f}(\bar{x}^0), \ldots, D_n\tilde{f}(\bar{x}^0)),$$

and it defines a fuzzy-valued function from X to $F^n(\mathbb{R}) = F(\mathbb{R}) \times \cdots \times F(\mathbb{R})$ (n times), where each $D_i\tilde{f}(\bar{x}^0)$ is a fuzzy number for $i = 1, \ldots, n$. The α-level set of $\nabla\tilde{f}(\bar{x}^0)$ is defined and denoted by

$$(\nabla\tilde{f}(\bar{x}^0))_\alpha = ((D_1\tilde{f}(\bar{x}^0))_\alpha, \ldots, (D_n\tilde{f}(\bar{x}^0))_\alpha),$$

where

$$(D_i\tilde{f}(\bar{x}^0))_\alpha = [D_i\tilde{f}_\alpha^L(\bar{x}^0), D_i\tilde{f}_\alpha^U(\bar{x}^0)],$$

$i = 1, \ldots, n$.

We say that $\tilde{f}$ is H-differentiable on X if it is H-differentiable at every $\bar{x}^0 \in X$.

Proposition 3.3. *Let X be an open subset of $\mathbb{R}^n$. If a fuzzy-valued function $\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable on X. Then $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are also differentiable on X, for all $\alpha \in [0,1]$. Moreover, for each $\bar{x} \in X$, $(D_i\tilde{f}(\bar{x}))_\alpha = [D_i\tilde{f}_\alpha^L(\bar{x}), D_i\tilde{f}_\alpha^U(\bar{x})]$, $i = 1, \ldots, n$.*

Proof: The result follows from Propositions 3.1 and 3.2.

Definition 3.7. *We say that $\tilde{f}$ is continuously H-differentiable at $\bar{x}^0$ if all of the partial H-derivatives $\partial\tilde{f}/\partial x_i$, $i = 1, \ldots, n$, exist on some neighborhoods of $\bar{x}^0$ and are continuous at $\bar{x}^0$ (in the sense of fuzzy-valued function). We say that $\tilde{f}$ is continuously H-differentiable on X if it is continuously H-differentiable at every $x^0 \in X$.*

Proposition 3.4. *Let $\tilde{f} : X \to F(\mathbb{R})$ be continuously H-differentiable on X. Then $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also continuously differentiable on X, for all $\alpha \in [0, 1]$.*

Proof: Follows by Propositions 3.1 and 3.3.

Now we define twice continuously H-differentiable fuzzy-valued function.

Definition 3.8. *Let $\tilde{f} : X \to F(\mathbb{R}), X \subset \mathbb{R}^n$ be a fuzzy-valued function. Suppose now that there is $\bar{x}^0 \in X$ such that gradient of $\tilde{f}$, $\nabla\tilde{f}$, is itself H-differentiable at $\bar{x}^0$, that is, for each i, the function $D_i\tilde{f} : X \to F(\mathbb{R})$ is H-differentiable at $\bar{x}^0$. Denote the H-partial derivative of $D_i\tilde{f}$ in the direction of $\bar{e}_j$ at $\bar{x}^0$ by*

$$D^2_{ij}\tilde{f} \text{ or } \frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_i \partial x_j}, \ if \ i \neq j,$$

and

$$D^2_{ii}\tilde{f} \text{ or } \frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_i^2}, \ if \ i = j.$$

Then we say that $\tilde{f}$ is twice H-differentiable at $\bar{x}^0$, with second H-derivative $\nabla^2\tilde{f}(\bar{x}^0)$ which can be called as the fuzzy Hessian matrix and is denoted by

$$\nabla^2\tilde{f}(\bar{x}^0) = \begin{pmatrix} \frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_1^2} & \dots & \frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_1 \partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_n \partial x_1} & \dots & \frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_n^2} \end{pmatrix}$$

where $\frac{\partial^2\tilde{f}(\bar{x}^0)}{\partial x_i \partial x_j} \in F(\mathbb{R})$, $i, j = 1, \dots, n$.
If $\tilde{f}$ is twice H-differentiable at each $\bar{x}^0$ in X, we say that $\tilde{f}$ is twice H-differentiable on X, and if for each $i, j = 1, \dots, n$, the cross-partial derivative $\frac{\partial^2\tilde{f}}{\partial x_i \partial x_j}$ is continuous function from X to $F(\mathbb{R})$, we say that $\tilde{f}$ is twice continuously H-differentiable on X.

Proposition 3.5. *Let $\tilde{f} : X \subseteq \mathbb{R}^n \to F(\mathbb{R})$ be H-differentiable with derivative $\nabla\tilde{f}$ on X and let each $D_i\tilde{f} : X \to F(\mathbb{R})$, $i = 1, \dots, n$, be also H-differentiable at $\bar{x}^0$ with H-derivative $D^2_{ij}\tilde{f}(\bar{x}^0)$, $i, j = 1, \dots, n$. Then $D_i\tilde{f}^L_\alpha$ and $D_i\tilde{f}^U_\alpha$ are also differentiable at $\bar{x}^0$, for all $\alpha \in [0, 1]$. Also, we have $(D^2_{ij}\tilde{f}(\bar{x}^0))_\alpha = [D^2_{ij}(\tilde{f}^L_\alpha)(\bar{x}^0), D^2_{ij}(\tilde{f}^U_\alpha)(\bar{x}^0)]$, $i, j = 1, \dots, n$.*

Proof: Follows by Proposition 3.4.

We define definiteness and semi-definiteness of a fuzzy matrix.

Definition 3.9. *Let $\tilde{A} = (\tilde{a}_{ij})$, $i, j = 1, \dots, n$ be a fuzzy matrix. That is, all the elements $(\tilde{a}_{ij})$ in the fuzzy matrix $\tilde{A}$, are fuzzy numbers defined on $\mathbb{R}$. There are associated two real matrices called α-level matrices, $\tilde{A}^L_\alpha$ and $\tilde{A}^U_\alpha$, $\alpha \in [0, 1]$ which are given as follows:*

$$\tilde{A}^L_\alpha = \begin{pmatrix} (\tilde{a}_{11})^L_\alpha & \dots & (\tilde{a}_{1n})^L_\alpha \\ \dots & \dots & \dots \\ (\tilde{a}_{n1})^L_\alpha & \dots & (\tilde{a}_{nn})^L_\alpha \end{pmatrix}$$

and

$$\tilde{A}^U_\alpha = \begin{pmatrix} (\tilde{a}_{11})^U_\alpha & \dots & (\tilde{a}_{1n})^U_\alpha \\ \dots & \dots & \dots \\ (\tilde{a}_{n1})^U_\alpha & \dots & (\tilde{a}_{nn})^U_\alpha \end{pmatrix}.$$

Then $\tilde{A}$ is said to be

(i) *positive definite fuzzy matrix if the α-level matrices $\tilde{A}^L_\alpha$ and $\tilde{A}^U_\alpha$ are positive definite real matrices, for all $\alpha \in [0, 1]$,*

(ii) *positive semidefinite fuzzy matrix if the α-level matrices $\tilde{A}^L_\alpha$ and $\tilde{A}^U_\alpha$ are positive semidefinite real matrices, for all $\alpha \in [0, 1]$.*

Example 3.1. Consider the fuzzy matrix

$$\tilde{A} = \begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{a} \end{pmatrix}$$

where $\tilde{a} = (1, 2, 4)$ and $\tilde{0} = (0, 0, 0)$ are fuzzy numbers. Then we obtain two α-level matrices for $\tilde{A}$,

$$\tilde{A}^L_\alpha = \begin{pmatrix} (1+\alpha) & 0 \\ 0 & (1+\alpha) \end{pmatrix} \textit{and } \tilde{A}^U_\alpha = \begin{pmatrix} (4-2\alpha) & 0 \\ 0 & (4-2\alpha) \end{pmatrix}$$

for all $\alpha \in [0, 1]$. Clearly these matrices are positive definite for all α. Therefore, the given fuzzy matrix $\tilde{A}$ is positive definite.

Example 3.2. Now consider the fuzzy matrix

$$\tilde{B} = \begin{pmatrix} \tilde{a} & \tilde{0} \\ \tilde{0} & \tilde{a} \end{pmatrix}$$

where $\tilde{a} = (0, 2, 4)$ and $\tilde{0} = (0, 0, 0)$ are fuzzy numbers. Then we obtain two α-level matrices for $\tilde{B}$,

$$\tilde{B}^L_\alpha = \begin{pmatrix} (2\alpha) & 0 \\ 0 & (2\alpha) \end{pmatrix} \textit{and } \tilde{B}^U_\alpha = \begin{pmatrix} (4-2\alpha) & 0 \\ 0 & (4-2\alpha) \end{pmatrix}$$

for all $\alpha \in [0, 1]$. These matrices are positive definite, for all α except $\alpha = 0$. For $\alpha = 0$, the first matrix is positive semidefinite. Therefore, the given fuzzy matrix $\tilde{B}$ is positive semidefinite.

4. Necessary and sufficient optimality conditions for unconstrained fuzzy optimization problem

4.1. **Problem and its solution.**

Let $T \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and $\tilde{f}$ be fuzzy-valued function defined on T. Consider the following nonlinear unconstrained fuzzy optimization problem (FOP).

$$\text{Minimize} \quad \tilde{f}(x) = \tilde{f}(x_1, .., x_n),$$
$$\text{subject to} \quad \bar{x} \in T.$$

A locally non-dominated solution of (FOP) is given as follows.

Definition 4.1. *Let T be an open subset of $\mathbb{R}^n$. A point $\bar{x}^0 \in T$ is a locally non-dominated solution of (FOP) if there exists no $\bar{x}^1 (\neq \bar{x}^0) \in N_\epsilon(\bar{x}^0) \cap T$ such that $\tilde{f}(\bar{x}^1) \preceq \tilde{f}(\bar{x}^0)$, where $N_\epsilon(\bar{x}^0)$ is a ϵ-neighborhood of $\bar{x}^0$.*

4.2. **Necessary and sufficient optimality conditions.**

The first and second order necessary and sufficient optimality conditions for real unconstrained optimization problem, given in [5], are as follows.

Theorem 4.1. *Let T be an open subset of $\mathbb{R}^n$.*

(i) (FONC) Let f be a continuously differentiable function on T. If x^* is a local minimizer of f over T, then $\nabla f(x^*) = 0$.

(ii) (SONC) Let f be a twice continuously differentiable function on T. If x^* is a local minimizer of f over T, then $\nabla^2 f(x^*)$ is positive semidefinite.

(iii) (SOSC) Let f be a twice continuously differentiable function on T. Suppose that
(1) $\nabla f(x^*) = 0$ and
(2) $\nabla^2 f(x^*)$ is positive definite.
Then x^* is a strict local minimizer of f.

We prove here necessary and sufficient optimality conditions for obtaining the locally non-dominated solution of (FOP). We need the following Theorem of classical optimization theory given in [1].

Theorem 4.2. [1] *Suppose that $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\bar{x}$. If there is a vector $\bar{d}$ such that $\nabla f(\bar{x})^T \cdot \bar{d} < 0$, then there exists a $\delta > 0$ such that $f(\bar{x} + \lambda \bar{d}) < f(\bar{x})$ for each $\lambda \in (0, \delta)$, so that $\bar{d}$ is a descent direction of f at $\bar{x}$.*

The first order necessary condition is as follows.

Theorem 4.3. *Suppose* $\tilde{f} : T \to F(\mathbb{R})$ *is continuously H-differentiable fuzzy-valued function,* T *is an open subset of* $\mathbb{R}^n$*. If* $\bar{x}^0 \in T$ *is a locally non-dominated solution of (FOP) and for any direction* $\bar{d}$ *and for any* $\delta > 0$ *there exists* $\lambda \in (0, \delta)$ *such that* $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$ *and* $\tilde{f}(\bar{x}^0)$ *are comparable, then* $\nabla \tilde{f}(\bar{x}^0) = \tilde{0}$.

Proof: Suppose that

$$\nabla \tilde{f}(\bar{x}^0) \neq \tilde{0},$$

then there exists $\alpha_0 \in [0, 1]$ such that

$$\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0) \neq 0$$

or

$$\nabla \tilde{f}^U_{\alpha_0}(\bar{x}^0) \neq 0.$$

Without loss of generality suppose that

$$\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0) \neq 0.$$

Let

$$\bar{d} = -\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0).$$

Then we get

$$\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0) \cdot \bar{d} = -\|\nabla \tilde{f}^L_{\alpha_0}(\bar{x}^0)\|^2 < 0.$$

By Theorem 4.2, there is a $\delta > 0$ such that

$$\tilde{f}^L_{\alpha_0}(\bar{x}^0 + \lambda \bar{d}) < \tilde{f}^L_{\alpha_0}(\bar{x}^0) \tag{4.1}$$

for $\lambda \in (0, \delta)$. Now by assumption of the theorem,
for any direction $\bar{d}$ and for any $\delta > 0$ there exists $\lambda \in (0, \delta)$ such that $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$ and $\tilde{f}(\bar{x}^0)$ are comparable.

Thus, either $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d}) \preceq \tilde{f}(\bar{x}^0)$ or $\tilde{f}(\bar{x}^0) \preceq \tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$. But from (4.1), we must have

$$\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d}) \prec \tilde{f}(\bar{x}^0).$$

Which contradicts to our assumption that $\bar{x}^0$ is a non-dominated solution. Therefore,

$$\nabla \tilde{f}(\bar{x}^0) = \tilde{0}.$$

Remark:

$$\nabla \tilde{f}(\bar{x}^0) = \tilde{0}$$

implies

$$\nabla \tilde{f}_\alpha^L(\bar{x}^0) = 0 \textit{ and } \nabla \tilde{f}_\alpha^U(\bar{x}^0) = 0$$

for all $\alpha \in [0,1]$. This implies

$$\int_0^1 \nabla \tilde{f}_\alpha^L(\bar{x}^0) \cdot d\alpha = 0 \textit{ and } \int_0^1 \nabla \tilde{f}_\alpha^U(\bar{x}^0) \cdot d\alpha = 0$$

That is

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = 0$$

Next, we prove second order necessary condition.

Theorem 4.4. *Let $\tilde{f}$ be a twice continuously H-differentiable fuzzy-valued function defined on $T \subseteq \mathbb{R}^n$. If $\bar{x}^0$ is a locally non-dominated solution of (FOP) and for any direction $\bar{d}$ and for any $\delta > 0$ there exists $\lambda \in (0,\delta)$ such that $\tilde{f}(\bar{x}^0 + \lambda \cdot \bar{d})$ and $\tilde{f}(\bar{x}^0)$ are comparable then $\nabla^2 \tilde{f}(\bar{x}^0)$ is positive semidefinite fuzzy matrix.*

Proof: We prove the result by contradiction. Suppose $\nabla^2 \tilde{f}(\bar{x}^0)$ is not a positive semidefinite fuzzy matrix. Then by definition, there exists some $\alpha_0 \in [0,1]$, such that either

$$\bar{d}_0^t \cdot \nabla^2 \tilde{f}_{\alpha_0}^L(\bar{x}) \cdot \bar{d}_0 < 0$$

or

$$\bar{d}_0^t \cdot \nabla^2 \tilde{f}_{\alpha_0}^U(\bar{x}) \cdot \bar{d}_0 < 0,$$

for some direction $\bar{d}_0$. Without loss of generality, we assume that

$$\bar{d}_0^t \cdot \nabla^2 \tilde{f}_{\alpha_0}^L(\bar{x}) \cdot \bar{d}_0 < 0 \tag{4.2}$$

Now let $\bar{x}(\beta) = \bar{x}^0 + \beta \bar{d}$ and define the composite function

$$\phi_\alpha(\beta) = \tilde{f}_\alpha^L(\bar{x}^0 + \beta \bar{d}),$$

for all $\alpha \in [0,1]$. Since $\tilde{f}$ is twice continuously H-differentiable fuzzy-valued function on T. By Proposition 3.1 and 3.5, $\tilde{f}_\alpha^L$ and $\tilde{f}_\alpha^U$ are also twice continuously differentiable functions on T, for all $\alpha \in [0,1]$. Then by Taylor's theorem,

$$\phi_\alpha(\beta) = \phi_\alpha(0) + \phi'_\alpha(0) \cdot \beta + \phi''_\alpha(0) \cdot \frac{\beta^2}{2} + O(\beta^2),$$

for all $\alpha \in [0,1]$. Now since $\bar{x}^0$ is a locally non-dominated solution of (FOP) then by Theorem 4.3,

$$\phi'_\alpha(0) = \bar{d} \cdot \nabla \tilde{f}_\alpha^L(\bar{x}^0) = 0,$$

for all $\alpha \in [0,1]$. Therefore,

$$\phi_\alpha(\beta) - \phi_\alpha(0) = \phi''_\alpha(0) \cdot \frac{\beta^2}{2} + O(\beta^2).$$

Since $\phi''_\alpha(0) = \bar{d}^t \cdot \nabla^2 \tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d}$,

$$\phi_\alpha(\beta) - \phi_\alpha(0) = (\bar{d}^t \cdot \nabla^2 \tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d})\frac{\beta^2}{2} + O(\beta^2).$$

Taking $\alpha = \alpha_0$ and $\bar{d} = \bar{d}_0$, from (4.2) and for sufficiently small β,

$$\phi_{\alpha_0}(\beta) - \phi_{\alpha_0}(0) < 0.$$

That is,

$$\tilde{f}^L_{\alpha_0}(\bar{x}^0 + \beta\bar{d}_0) < \tilde{f}^L_{\alpha_0}(\bar{x}^0) \tag{4.3}$$

Now β is chosen in such a way that $\tilde{f}(\bar{x}^0+\beta\bar{d}_0)$ and $\tilde{f}(\bar{x}^0)$ are comparable. That is either $\tilde{f}(\bar{x}^0+\beta\bar{d}_0) \preceq \tilde{f}(\bar{x}^0)$ or $\tilde{f}(\bar{x}^0+\beta\bar{d}_0) \succeq \tilde{f}(\bar{x}^0)$. But $\tilde{f}(\bar{x}^0+\beta\bar{d}_0) \succeq \tilde{f}(\bar{x}^0)$ not possible because of (4.3). Therefore,

$$\tilde{f}(\bar{x}^0 + \beta\bar{d}_0) \preceq \tilde{f}(\bar{x}^0)$$

which contradicts the assumption that , $\bar{x}^0$ is a locally non-dominated solution. Therefore, $\nabla^2\tilde{f}(\bar{x}^0)$ is a positive semidefinite fuzzy matrix.

Now, we prove second-order sufficient condition.

Theorem 4.5. *Let $\tilde{f}$ be a twice continuously H-differentiable function on $T \subseteq \mathbb{R}^n$. Suppose that*

(1) $\nabla\tilde{f}(\bar{x}^0) = \tilde{0}$

(2) $\nabla^2\tilde{f}(\bar{x}^0)$ *is positive definite fuzzy matrix.*

Then, $\bar{x}^0$ is locally non-dominated solution of (FOP).

Proof: We prove this result by contradiction. Suppose $\bar{x}^0 \in T$ is not a locally non-dominated solution of (FOP). Then, for any $\epsilon > 0$ there exists $\bar{x}^1(\neq \bar{x}^0) \in N_\epsilon(\bar{x}^0) \cap T$ such that $\tilde{f}(\bar{x}^1) \preceq \tilde{f}(\bar{x}^0)$. That is., there exists $\bar{x}^1 \in N_\epsilon(\bar{x}^0) \cap T$ such that

$$\tilde{f}(\bar{x}^1)^L_\alpha \leq \tilde{f}(\bar{x}^0)^L_\alpha \textit{ and } \tilde{f}(\bar{x}^1)^U_\alpha \leq \tilde{f}(\bar{x}^0)^U_\alpha \tag{4.4}$$

for all $\alpha \in [0,1]$. Now since $\tilde{f}$ is the twice continuously H-differentiable function, $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ are also twice continuously differentiable functions, for all $\alpha \in [0.1]$. Using assumption 2 and Rayleigh's inequality (refer [4], page no. 34), it follows that if $\bar{d} \neq 0$, then

$$0 < \lambda_{\min}(\nabla^2\tilde{f}^L_\alpha(\bar{x}^0))\|\bar{d}\|^2 \leq \bar{d}^t \cdot \nabla^2\tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d}.$$

By Taylor's theorem and assumption 1,

$$\begin{aligned}
\tilde{f}^L_\alpha(\bar{x}^0 + \bar{d}) - \tilde{f}^L_\alpha(\bar{x}^0) &= \frac{1}{2}\bar{d}^t \cdot \nabla^2\tilde{f}^L_\alpha(\bar{x}^0) \cdot \bar{d} + O(\|\bar{d}\|^2) \\
&\geq \frac{\lambda_{\min}(\nabla^2\tilde{f}^L_\alpha(\bar{x}^0))}{2}\|\bar{d}\|^2 + O(\|\bar{d}\|^2) \\
&> 0,
\end{aligned}$$

for all $\bar{d}$ such that $\|\bar{d}\|$ is sufficiently small. Now choose $\bar{x}^1$ so close to $\bar{x}^0$ so that $\bar{d} = \bar{x}^1 - \bar{x}^0$ is sufficiently small and hence,

$$\tilde{f}^L_\alpha(\bar{x}^1) - \tilde{f}^L_\alpha(\bar{x}^0) = \tilde{f}^L_\alpha(\bar{x}^0 + \bar{d}) - \tilde{f}^L_\alpha(\bar{x}^0) > 0$$

That is,

$$\tilde{f}^L_\alpha(\bar{x}^1) > \tilde{f}^L_\alpha(\bar{x}^0)$$

This gives contradiction to inequality (4.4). Hence proved the result.

We consider two examples to illustrate the above results.

Example 4.1.

$$\text{Minimize} \qquad \tilde{f}(x_1, x_2),$$
$$\text{subject to} \quad \bar{x} = (x_1, x_2) \in \mathbb{R}^2,$$

where $\tilde{f} : \mathbb{R}^2 \to F(\mathbb{R})$ be defined by

$$\tilde{f}(x_1, x_2) = (1, 2, 4) \odot x_1^2 \oplus (1, 2, 4) \odot x_2^2 \oplus (1, 3, 5),$$

$(1, 2, 4)$ and $(1, 3, 5)$ are triangular fuzzy numbers.
By the first order necessary condition, we have

$$\int_0^1 \{\nabla \tilde{f}^L_\alpha(\bar{x}^0) + \nabla \tilde{f}^U_\alpha(\bar{x}^0)\} \cdot d\alpha = 0$$

Here, $\tilde{f}^L_\alpha(x_1, x_2) = (1 + \alpha)x_1^2 + (1 + \alpha)x_2^2 + (1 + 2\alpha)$ and
$\tilde{f}^U_\alpha(x_1, x_2) = (4 - 2\alpha)x_1^2 + (4 - 2\alpha)x_2^2 + (5 - 2\alpha)$.
Therefore,

$$\nabla f^L_\alpha(x_1, x_2) = \begin{pmatrix} 2(1+\alpha)x_1 \\ 2(1+\alpha)x_2 \end{pmatrix} \textit{and } \nabla f^U_\alpha(x_1, x_2) = \begin{pmatrix} 2(4-2\alpha)x_1 \\ 2(4-2\alpha)x_2 \end{pmatrix}.$$

Therefore,

$$\int_0^1 \{\nabla \tilde{f}^L_\alpha(\bar{x}^0) + \nabla \tilde{f}^U_\alpha(\bar{x}^0)\} \cdot d\alpha = \begin{pmatrix} 9x_1 \\ 9x_2 \end{pmatrix} = 0.$$

That is, $x^0 = (x_1, x_2) = (0, 0)$.
Now to verify second order necessary and sufficient conditions, we find fuzzy Hessian matrix of $\tilde{f}(x)$. The α-level matrices of fuzzy Hessian matrix are

$$\nabla^2 \tilde{f}^L_\alpha(x) = \begin{pmatrix} 2(1+\alpha) & 0 \\ 0 & 2(1+\alpha) \end{pmatrix}$$

and

$$\nabla^2 \tilde{f}^U_\alpha(x) = \begin{pmatrix} 2(4-2\alpha) & 0 \\ 0 & 2(4-2\alpha) \end{pmatrix}.$$

Since both the α-level matrices $\nabla^2 \tilde{f}^L_\alpha(x)$ and $\nabla^2 \tilde{f}^U_\alpha(x)$ are positive definite matrices for all $\alpha \in [0, 1]$. Therefore, $x^0 = (0, 0)$ satisfies the second order necessary and sufficient condition for a locally non-dominated solution. Hence, $x^0 = (0,0)$ is a locally non-dominated solution of given problem.

Now we consider another example.

Example 4.2.

$$\begin{aligned} &Minimize \quad \tilde{f}(x_1, x_2), \\ &subject\,to \quad \bar{x} = (x_1, x_2) \in \mathbb{R}^2, \end{aligned}$$

where $\tilde{f} : \mathbb{R}^2 \to F(\mathbb{R})$ be defined by

$$\tilde{f}(x_1, x_2) = (1,2,4) \odot x_1^3 \oplus (1,2,4) \odot x_2^3 \oplus (1,3,5),$$

$(1,2,4)$ and $(1,3,5)$ are triangular fuzzy numbers.
By the first order necessary condition, we have

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = 0.$$

Here, $\tilde{f}_\alpha^L(x_1, x_2) = (1+\alpha)x_1^3 + (1+\alpha)x_2^3 + (1+2\alpha)$ and
$\tilde{f}_\alpha^U(x_1, x_2) = (4-2\alpha)x_1^3 + (4-2\alpha)x_2^3 + (5-2\alpha)$.
Therefore,

$$\nabla f_\alpha^L(x_1, x_2) = \begin{pmatrix} 3(1+\alpha)x_1^2 \\ 3(1+\alpha)x_2^2 \end{pmatrix}$$

and

$$\nabla f_\alpha^U(x_1, x_2) = \begin{pmatrix} 3(4-2\alpha)x_1^2 \\ 3(4-2\alpha)x_2^2 \end{pmatrix}.$$

Therefore,

$$\int_0^1 \{\nabla \tilde{f}_\alpha^L(\bar{x}^0) + \nabla \tilde{f}_\alpha^U(\bar{x}^0)\} \cdot d\alpha = \begin{pmatrix} 13.5x_1^2 \\ 13.5x_2^2 \end{pmatrix} = 0.$$

That is, $x^0 = (x_1, x_2) = (0,0)$.
Now to verify second order necessary and sufficient conditions, we find fuzzy Hessian matrix of $\tilde{f}(x)$. The α-level matrices of fuzzy Hessian matrix are:

$$\nabla^2 \tilde{f}_\alpha^L(x) = \begin{pmatrix} 6(1+\alpha)x_1 & 0 \\ 0 & 6(1+\alpha)x_2 \end{pmatrix}$$

and

$$\nabla^2 \tilde{f}_\alpha^U(x) = \begin{pmatrix} 6(4-2\alpha)x_1 & 0 \\ 0 & 6(4-2\alpha)x_2 \end{pmatrix}.$$

Since both the α-level matrices $\nabla^2 \tilde{f}_\alpha^L(x)$ and $\nabla^2 \tilde{f}_\alpha^U(x)$ are positive semidefinite matrices for all $\alpha \in [0,1]$ at point $x^0 = (0,0)$. Therefore, $x^0 = (0,0)$ satisfies the second order necessary condition but not the sufficient condition for a locally non-dominated solution, as none of the α-level matrices is positive definite at $x^0 = (0, 0)$. Hence, $x^0 = (0, 0)$ is not a locally non-dominated solution of given problem.

5. Conclusions

Using partial order relation on set of fuzzy numbers, the first order necessary and sufficient optimality conditions for a nonlinear unconstrained fuzzy optimization problem have been derived in this paper. We have used Hukuhara differentiability of a fuzzy-valued function for proving the same. We have also provided two examples to illustrate the possible applications in this subject.

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