Int. J. Adv. Appl. Math. and Mech. 3(1) (2015) 87 – 91 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Solution of fuzzy heat equations using Adomian Decomposition method

Research Article

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Received 11 July 2015; accepted (in revised version) 13 August 2015

Abstract: Solution of homogeneous fuzzy partial differential equations with specific fuzzy boundary and initial conditions are proposed. Using Adomian Decomposition method, we solve the heat equations for which it is difficult to find the solution by classical methods. We extend the crisp solution in the fuzzy form as a Seikkala solution.

MSC: 34A07 • 35K51

Keywords: Fuzzy heat equation • Seikkala solution • Adomian Decomposition method

1. Introduction

Study of fuzzy partial differential equations (FPDEs) means the generalization of partial differential equations (PDEs) in fuzzy sense. While doing modelling of real situation in terms of partial differential equation, we see that the variables and parameters involve in the equations are uncertain (in the sense that they are not completely known or inexact or imprecise). Many times common initial or boundary condition of ambient temperature is a fuzzy condition since ambient temperature is prone to variation in a range. We express this impreciseness and uncertainties in terms of fuzzy numbers. So we come across with fuzzy partial differential equations . In [4], Buckley and Feuring (1999) proposed a procedure to examine solutions of elementary fuzzy partial differential equations. First they verified the Buckley - Feuring (BF) solution exist or not. If the BF-solution fails to exists they looked for the Seikkala solution. Seikkala solution is based on Seikkala derivative introduced in [12]. Their proposed method works for elementary fuzzy partial differential equations. They assumed the solution of FPDE is not defined in terms of series.

The Adomian decomposition method was introduced and developed by George Adomian in [1, 2] and is well addressed in the literature. A considerable amount of research work has been invested in applying this method to a wide class of linear and non-linear ordinary differential equations, partial differential equations and integral equations as well. For more details, see the references [1, 2], [5, 6], [13–16] and the references therein. Some recent literatures on Modified Adomian Decompositions methods and Laplace Decomposition methods are found in [9], [10] and [11]. Babolian at el. (2004)[3] have proposed the numerical approximate solution of first order fuzzy initial value problem using Adomian method.

In this paper, we find fuzzy solutions of second order homogeneous fuzzy partial differential equations. Using Seikkala derivative, we solve a fuzzy heat equation with specific fuzzy boundary and initial conditions. We solve the crisp form of heat equation using Adomian Decomposition method and then extend the solution in fuzzy form as a Seikkala solution.

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2. Fuzzy numbers

We start with some basic definitions.

Definition 2.1 ([7]).
Let $\mathbb{R}$ be the set of real numbers and $\tilde{a} : \mathbb{R} \to [0,1]$ be a fuzzy set. We say that $\tilde{a}$ is a fuzzy number if it satisfies the following properties:

(i) $\tilde{a}$ is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $\tilde{a}(x_0) = 1$;

(ii) $\tilde{a}$ is fuzzy convex, that is, $\tilde{a}(tx + (1-t)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$, whenever x, y $\in \mathbb{R}$ and $t \in [0,1]$;

(iii) $\tilde{a}(x)$ is upper semi-continuous on $\mathbb{R}$, that is, $\{x/\tilde{a}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0,1]$;

(iv) $cl\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$ forms a compact set,

where cl denotes closure of a set. The set of all fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$. For all $\alpha \in (0,1]$, α-level set $\tilde{a}_\alpha$ of any $\tilde{a} \in F(\mathbb{R})$ is defined as $\tilde{a}_\alpha = \{x \in \mathbb{R}/\tilde{a}(x) \geq \alpha\}$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R}/\tilde{a}(x) > 0\}$. By definition of fuzzy numbers, we can prove that, for any $\tilde{a} \in F(\mathbb{R})$ and for each $\alpha \in (0,1]$, $\tilde{a}_\alpha$ is compact convex subset of $\mathbb{R}$, and we write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. $\tilde{a} \in F(\mathbb{R})$ can be recovered from its α-level sets by a well-known decomposition theorem (ref. [8]), which states that $\tilde{a} = \cup_{\alpha \in [0,1]} \alpha \cdot \tilde{a}_\alpha$ where union on the right-hand side is the standard fuzzy union.

Definition 2.2.
According to Zadeh's extension principle, we define addition of two fuzzy numbers $\tilde{a}, \tilde{b}$ and scalar multiplication of fuzzy number $\tilde{a}$ with a scalar $\lambda \in \mathbb{R}$ by their α-level sets as follows:

$$
\begin{aligned}
(\tilde{a} \oplus \tilde{b})_\alpha &= [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha] \\
(\lambda \odot \tilde{a})_\alpha &= [\lambda \cdot \tilde{a}^L_\alpha, \lambda \cdot \tilde{a}^U_\alpha],\ if\ \lambda \geq 0 \\
&= [\lambda \cdot \tilde{a}^U_\alpha, \lambda \cdot \tilde{a}^L_\alpha],\ if\ \lambda < 0,
\end{aligned}
$$

where α-level sets of $\tilde{a}$ and $\tilde{b}$ are $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$, $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ for $\alpha \in [0,1]$.

3. Seikkala solution (S-solution) of fuzzy partial differential equation

We study the homogeneous partial differential equation in the form

$$\phi(U(x,t)) = 0, \tag{1}$$

where ϕ is a linear partial derivative operator. We find the solution of the problem in crisp sense and then extend the solution for the fuzzy problem. Homogeneous fuzzy partial differential equation associated with (1) is

$$\phi(\tilde{U}(x,t)) = \tilde{0}, \tag{2}$$

where $\tilde{0}(r) = 1$ if $r = 0$ and $\tilde{0}(r) = 0$ if $r \neq 0$, subject to certain fuzzy boundary and initial conditions. We find the Seikkala solution of (2) and hence find the solution of fuzzy heat equation subject to specific fuzzy boundary and initial conditions.

Seikkala differentiability of fuzzy-valued function $\tilde{U}(x,t)$ is defined by Seikkala [12].

Definition 3.1 ([12]).
Let $\tilde{U}_\alpha(x,t) = [u_1(x,t,\alpha), u_2(x,t,\alpha)]$, for all α. We assume that $u_i(x,t,\alpha)$ have continuous partial derivatives so that $\phi(u_i(x,t,\alpha))$ is continuous for all $(x,t) \in I_1 \times I_2$ and all α, $i = 1,2$. Define

$$(DU)_\alpha(x,t) = [\phi(u_1(x,t,\alpha)), \phi(u_2(x,t,\alpha))]$$

for all $(x,t) \in I_1 \times I_2$ and all α. If, for each fixed $(x,t) \in I_1 \times I_2$, $(DU)_\alpha(x,t)$ defines the α-level set of a fuzzy number, then we will say that $\tilde{U}(x,t)$ is differentiable and we write

$$\phi(\tilde{U}_\alpha(x,t)) = (DU)_\alpha(x,t)$$

for all $(x,t) \in I_1 \times I_2$ and all α.

The sufficient conditions for $(DU)_\alpha(x,t)$ to define α-level sets of a fuzzy number are

(i) $\phi(u_1(x,t,\alpha))$ is an increasing function of α for each $(t,x) \in I_1 \times I_2$;

(ii) $\phi(u_2(x,t,\alpha))$ is a decreasing function of α for each $(t,x) \in I_1 \times I_2$; and

(iii) $\phi(u_1(x,t,1)) \leq \phi(u_2(x,t,1))$ for all $(x,t) \in I_1 \times I_2$.

S-solution of problem (2) is $\tilde{U}(x,t)$ if Seikkala derivative exists and satisfies equation (2). Let $\tilde{U}_\alpha(x,t) = [u_1(x,t,\alpha), u_2(x,t,\alpha)]$.

Consider the system of partial differential equations

$$\phi(u_1(x,t,\alpha)) = 0, \tag{3}$$

$$\phi(u_2(x,t,\alpha)) = 0 \tag{4}$$

for all $(x,t) \in I_1 \times I_2$ and all $\alpha \in [0,1]$. The fuzzy boundary conditions are $\tilde{U}(0,t) = \tilde{C}_1$ and $\tilde{U}(M_1,t) = \tilde{C}_2$ and fuzzy initial condition is $\tilde{U}(x,0) = \tilde{f}(x)$, where $\tilde{C}_1$, $\tilde{C}_2$ are fuzzy numbers and $\tilde{f}(x)$ is a fuzzy-valued function of x. We write boundary conditions in terms of α-level sets as

$$u_1(0,t,\alpha) = c_{11}(\alpha),\ u_2(0,t,\alpha) = c_{12}(\alpha) \tag{5}$$

$$u_1(M_1,t,\alpha) = c_{21}(\alpha),\ u_2(M_1,t,\alpha) = c_{22}(\alpha). \tag{6}$$

The initial condition

$$u_1(x,0,\alpha) = \tilde{f}_1(x,\alpha),\ u_2(x,0,\alpha) = \tilde{f}_2(x,\alpha). \tag{7}$$

Let $u_i(x,t,\alpha)$ solve equations (3) and (4) with boundary conditions (5) and (6) and initial conditions (7) , $i = 1,2$. If

$$[u_1(x,t,\alpha), u_2(x,t,\alpha)] \tag{8}$$

defines the α-level set of a fuzzy number, for each $(x,t) \in I_1 \times I_2$, then $\tilde{U}(x,t)$ is the S-solution.

4. Adomian Decomposition method

Consider the following form of heat equation with given initial condition:

$$U_t = U_{xx},\ 0 < x < \pi,\ t > 0 \tag{9}$$

with initial condition $U(x,0) = f(x)$, $0 \leq x \leq \pi$. To solve heat equation (9) using Adomian Decomposition method, we rewrite (9) in an operator form.

$$L_t U = L_x U, \tag{10}$$

where $L_t = \dfrac{\partial}{\partial t}$, $L_x = \dfrac{\partial^2}{\partial x^2}$ with inverse operator

$$L_t^{-1} = \int_0^t (.)dt.$$

By applying L_t^{-1} both sides of (10) and using initial condition

$$U(x,t) = f(x) + L_t^{-1}(L_x(U(x,t))). \tag{11}$$

The Adomian decomposition method defines the unknown function $U(x,t)$ into a sum of components defined by series

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t), \tag{12}$$

where the components U_0, U_1, U_2,... are to be determined. Substituting equation (12) into both sides of (11),

$$\sum_{n=0}^{\infty} U_n(x,t) = f(x) + L_t^{-1}(L_x(\sum_{n=0}^{\infty} U_n(x,t))). \tag{13}$$

We get the recurrence scheme for the complete determination of the components $U_n(x,t)$, $n \geq 0$.

$$U_0(x,t) = f(x), \qquad U_{k+1}(x,t) = L_t^{-1}(L_x(U_k(x,t))),\ k \geq 0. \tag{14}$$

5. Solution of fuzzy heat equations

In this section, we find the solution of fuzzy heat equations.

Example 5.1.
A heat equation considered is

$$U_t = U_{xx}, \tag{15}$$

with initial condition $U(x,0) = c(x+\cos(x))$. The values of x and t are in $I_1 = [0,\pi/2]$ and $I_2 = [0,1]$ respectively.
The solution of (15) with the given initial conditions by Adomian decomposition method is given as

$$U(x,t) = c(x+e^{-t}\cos(x)). \tag{16}$$

Now we fuzzify the partial differential equation (15) as

$$\phi(\tilde{U}(x,t)) = \tilde{0}, \tag{17}$$

where $\phi = D_t - D_x D_x$. Fuzzy initial condition is $\tilde{U}(x,0) = \tilde{c}\odot(x+\cos(x))$, where $\tilde{c}$ is a fuzzy number (An operator $\odot$ defines multiplication of a fuzzy number with a real number).
A fuzzified form of solution (16) is

$$\tilde{U}(x,t) = \tilde{c}\odot(x+e^{-t}\cos(x)). \tag{18}$$

We can check that (18) is a S-solution of FPDE (17). Let $\tilde{U}_\alpha(x,t) = [u_1(x,t,\alpha), u_2(x,t,\alpha)]$. Consider the system of partial differential equations

$$\phi(u_1(x,t,\alpha)) = 0, \tag{19}$$
$$\phi(u_2(x,t,\alpha)) = 0 \tag{20}$$

for all $(x,t)\in I_1\times I_2$ and all $\alpha\in[0,1]$. We write the initial condition as

$$u_1(x,0,\alpha) = c_1(\alpha)(x+\cos(x)),\ u_2(x,0,\alpha) = c_2(\alpha)(x+\cos(x)). \tag{21}$$

Let $u_i(x,t,\alpha)$ solve equations (19) and (20) with initial conditions (21) , $i=1,2$. The solution is

$$u_i(x,t,\alpha) = c_i(\alpha)(x+e^{-t}\cos(x)), \tag{22}$$

for $i=1,2$. If

$$[u_1(x,t,\alpha), u_2(x,t,\alpha)] \tag{23}$$

defines the α-level set of a fuzzy number, for each $(x,t)\in I_1\times I_2$, then $\tilde{U}(x,t)$ is the S-solution. It can be seen easily that $[u_1(x,t,\alpha), u_2(x,t,\alpha)]$, for each $(x,t)\in I_1\times I_2$ and $\alpha\in[0,1]$ defines a fuzzy number which is a required solution as given in (18).

Example 5.2.
We consider an another example of heat equation which is not solvable by separation of variables method.

$$U_t = U_{xx} - U, \tag{24}$$

with initial condition $U(x,0) = c\sin(x)$. The values of x and t are in $I_1 = [0,\pi]$ and $I_2 = [0,\pi]$ respectively.
The solution of (24) with the given initial conditions by Adomian decomposition method is given as

$$U(x,t) = c(e^{-2t}\sin(x)). \tag{25}$$

Now we fuzzify the partial differential equation (24) as

$$\phi(\tilde{U}(x,t)) = \tilde{0}, \tag{26}$$

where $\phi = D_t - D_x D_x + 1$. Fuzzy initial condition is $\tilde{U}(x,0) = \tilde{c}\odot(\sin(x))$, where $\tilde{c}$ is a fuzzy number (An operator $\odot$ defines multiplication of a fuzzy number with a real number).
A fuzzified form of solution (25) is

$$\tilde{U}(x,t) = \tilde{c}\odot(e^{-2t}\sin(x)). \tag{27}$$

We can check that (27) is a S-solution of FPDE (26). Let $\tilde{U}_\alpha(x,t) = [u_1(x,t,\alpha), u_2(x,t,\alpha)]$. Consider the system of partial differential equations

$$\phi(u_1(x,t,\alpha)) = 0, \tag{28}$$

$$\phi(u_2(x,t,\alpha)) = 0 \tag{29}$$

for all $(x,t) \in I_1 \times I_2$ and all $\alpha \in [0,1]$. We write the initial condition as

$$u_1(x,0,\alpha) = c_1(\alpha)(\sin(x)),\ u_2(x,0,\alpha) = c_2(\alpha)(\sin(x)). \tag{30}$$

Let $u_i(x,t,\alpha)$ solve equations (28) and (29) with initial conditions (30) , $i = 1,2$. The solution is

$$u_i(x,t,\alpha) = c_i(\alpha)(e^{-2t}\sin(x)), \tag{31}$$

for $i = 1,2$. If

$$[u_1(x,t,\alpha), u_2(x,t,\alpha)] \tag{32}$$

defines the α-level set of a fuzzy number, for each $(x,t) \in I_1 \times I_2$, then $\tilde{U}(x,t)$ is the S-solution. It can be seen easily that $[u_1(x,t,\alpha), u_2(x,t,\alpha)]$, for each $(x,t) \in I_1 \times I_2$ and $\alpha \in [0,1]$ defines a fuzzy number which is a required solution as given in (27).

6. Conclusions

We successfully attempted the solution of fuzzy heat equations using Adomian Decomposition method. We proposed the fuzzy solution of the fuzzy heat equations as Seikkala solution.

Acknowledgments

This research work is supported by National Board for Higher Mathematics (NBHM), Department of Atomic Energy (DAE), India.

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